On the space of subgroups of generalized Baumslag-Solitar groups

Thesis of Sasha Bontemps, under the supervision of Damien Gaboriau

1st July 2026

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$$G \curvearrowright \mathrm{Sub}(G) \text{ by conjugation.}$$

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  for every finite subsets $I, O \subseteq G$.
- Topology of the pointwise convergence: $H_n \to_{n\to\infty} H$ iff for every $h \in H$ (*resp.* $h \notin H$), there exists $n_0 \in \mathbb{N}$ such that $h \in H_n$ (*resp.* $h \notin H$) for every $n > n_0$.

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Motivations and setting
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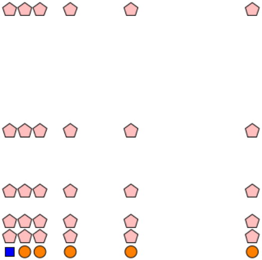

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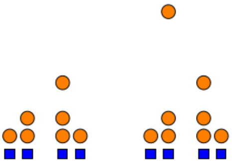

Theorem (Cantor–Bendixson)

There exists a unique partition $\mathrm{Sub}(G) = \mathcal{K}(G) \sqcup C$ into

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- a closed subset without isolated point $\mathcal{K}(G)$ called the **perfect kernel** of G.

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- $\mathcal{K}(\mathbb{F}_r) = \mathrm{Sub}_{[\infty]}(\mathbb{F}_r)$;
- [Bowen–Grigorchuk–Kravchenko] the perfect kernel of a lamplighter group is the set of lamp subgroups.

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- this is where are supported the "interesting" IRS's;
- $G \curvearrowright \mathcal{K}(G)$ contains the essential part of the dynamics.

An action $G \curvearrowright X$ of a group G on a topological space X by homeomorphisms is

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- [Hull–Minasyan–Osin, 2024] topologically μ-mixing for some $\mu \in \mathrm{Prob}(G)$ if, denoting by $(G_k)_k$ a sequence of iid random variables with law μ, for any two non-empty open subsets $U, V \subseteq X$, one has $\mathbb{P}\left((G_1 \ldots G_k \cdot U) \cap V \neq \emptyset\right) \to_{k\to\infty} 1$.

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for G in a particular class of groups...

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MR0142635 (26 204) 20.99
Baumslag, Gilbert; Solitar, Donald
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Bull. Amer. Math. Soc. **68** (1962), 199–201.
The group G with two generators a and b and the single defining relation $a^{-1}b^l a = b^m$ is a non-Hopf group, that is to say, isomorphic to a proper factor group of itself, unless either l divides m or m divides l or both have the same prime divisors. Thus in particular the single relation $a^{-1}b^2a = b^3$ defines a non-Hopf group. One of the consequences is that this group has a presentation with two generators and 1 relation, but in terms of two different generators, namely, a and b^4, it requires at least two defining relations. These facts destroy several conjectures long and strongly held by the reviewer and others, and some published and unpublished "theorems". Further results include the existence of a group (of the type described above, with $l = 12$ and $m = 18$) which is a Hopf group but has a non-Hopf normal subgroup of finite index; the existence of two non-isomorphic groups that are epimorphic images of each other, one being again defined by a single relation; and the existence of 2-generator non-Hopf groups that are soluble of length 3. Proofs are not given, except for one brief sketch. {Reference 5 is erroneously listed as by A. I. Mal′cev, but in the text it is correctly ascribed to W. Magnus.} *B. H. Neumann*

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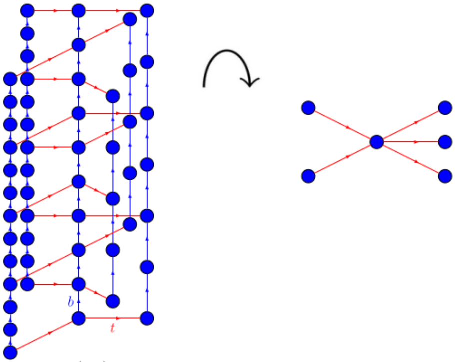

BS(2,3)

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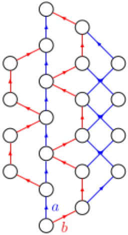

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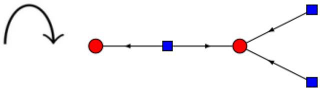

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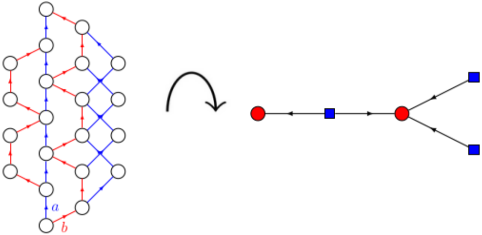

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In higher rank: semi-direct products $\mathbb{Z}^d \rtimes \mathbb{F}_r$, Leary-Minasyan groups...

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Definition

A GBS group G is **unimodular** if $|\det \circ \Delta| = 1$.

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is topologically μ-mixing.

$G = \mathrm{BS}(m,n)$ $(|m|,|n|>1)$. $\mathcal{T}_{m,n}$ Bass-Serre tree.

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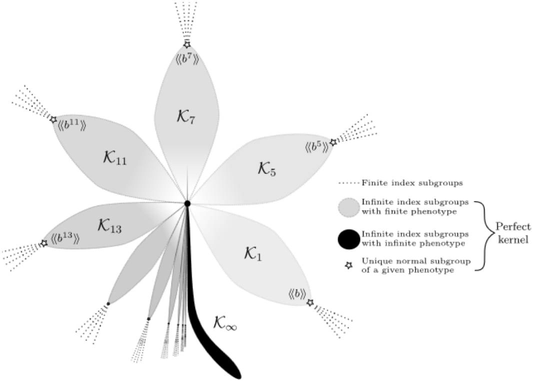
Groups acting on trees: state of the art

Baumslag-Solitar groups



Source: Carderi–Gaboriau–Le Maître–Stalder, *On the space of subgroups of Baumslag-Solitar groups I: perfect kernel and phenotype.*

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Theorem (B., 2025)

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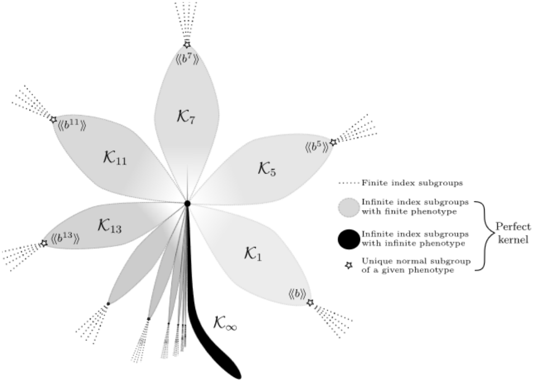

Source: Carderi–Gaboriau–Le Maître–Stalder, *On the space of subgroups of Baumslag-Solitar groups I: perfect kernel and phenotype.*

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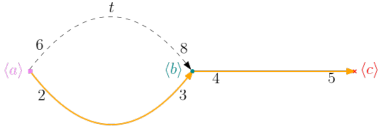
Question:

Is $G \curvearrowright \mathcal{K}_N$ HTT if $d > 1$?

An example:



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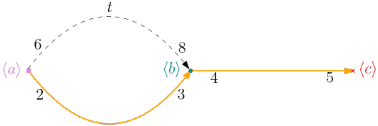

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Contributions of this thesis

Generalized Baumslag-Solitar groups: perfect kernel and dynamics

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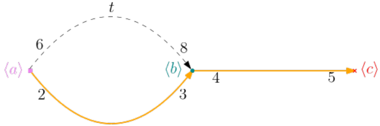

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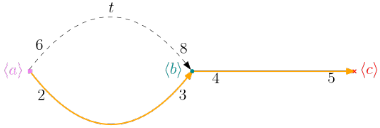

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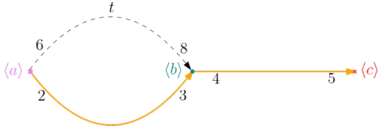

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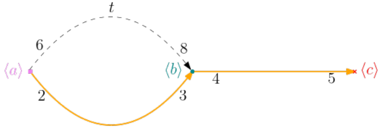

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If $P \in \mathrm{Ph}(\mathbb{N}^*)$, then

$$\mathcal{K}_P = \left\{ H \in \mathcal{K}(G) \mid H \cap \langle c \rangle = \left\langle c^N \right\rangle \text{ for some } N \in \mathrm{Ph}^{-1}(P) \right\}.$$

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$\mathcal{A} = \{G$ admitting an amenable, faithful and transitive action.$\}$

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Definition

Let $H \leq G$. The pair (G, H) has **relative property** (T) if for any unitary representation of G that almost has invariant vectors, there exists a unit H-invariant vector.

Dynamical characterization [Le Maître]: $G \in \mathcal{A}$ iff there exists a conjugacy class whose closure contains both a co-amenable subgroup, and a subgroup H that satisfies $\bigcap_{g\in G} gHg^{-1} = 1$.

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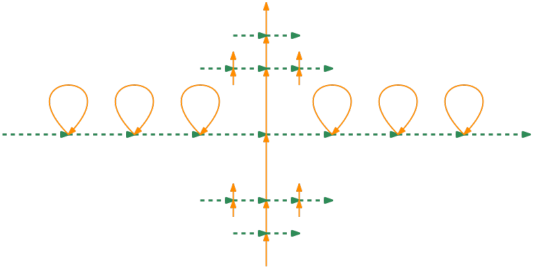

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Theorem (B., 2026)

Let G be a 2-GBS group. Then

$$\begin{aligned} G \notin \mathcal{A} &\iff (G, H) \text{ has } (T) \text{ for some } H \simeq \mathbb{Z}^2 \lhd G \\ &\iff G \text{ is virtually } \mathbb{Z}^2 \rtimes_\rho \mathbb{F}_r \text{ with } \rho(\mathbb{F}_r) \text{ non-virtually cyclic.} \end{aligned}$$

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–> Work in progress with Yves Cornulier.