SUBGROUP MIXING IN BAUMSLAG-SOLITAR GROUPS

SASHA BONTEMPS

ABSTRACT. In this article, we contribute to the study of the dynamics induced by the conjugation action on the space of subgroups of Baumslag-Solitar groups $\mathrm{BS}(m,n)$, via the mixing properties of elements asymptotically produced by suitable random walks on the group. In an acylindrically hyperbolic context, the authors of [HMO24] demonstrated strong mixing situations, namely topological μ-mixing, a strengthening of high topological transitivity. Regarding non-metabelian $\mathrm{BS}(m,n)$ with $|m| \neq |n|$, we exhibit here a radically different situation on each of the pieces except one of the partition introduced in [CGLMS22], (although it is highly topologically transitive on each piece). On the other hand, when $|m| = |n|$, we demonstrate the topological μ-mixing character on each of the pieces.

Keywords: Generalized Baumslag-Solitar groups; space of subgroups; Schreier graphs; perfect kernel; highly topologically transitive actions; topologically μ-mixing actions; random walks; Bass-Serre theory.

MSC-classification: 37B; 20E06; 20E08; 60G50.

1. Introduction

Given a couple $(m,n) \in \mathbb{Z}^*$, the Baumslag-Solitar group of parameter (m,n) is the group defined by the following presentation

$$\mathrm{BS}(m,n) = \langle b,t \mid tb^m t^{-1} = b^n \rangle. \tag{1}$$

Baumslag-Solitar groups were introduced in [BS62] to give the first examples of two-generated, finitely presented non Hopfian groups. They have been widely studied in relation to various properties, that strongly depend on the parameters (m,n): their residual finiteness ([Mes72]), their classification up to quasi-isometry (see [FM198] and [Why01]), their classification up to measure equivalence (announced by the authors of [GPT+])...The group $\mathrm{BS}(m,n)$ acts on its **Bass-Serre tree** $\mathcal{T}_{m,n}$ (*i.e.* the infinite oriented tree all of whose vertices have m incoming edges and n outgoing edges), with a single orbit of vertices and a single orbit of edges, and the vertex and edge stabilizers are infinite cyclic.

In this article, we pursue the study of the space of subgroups of Baumslag-Solitar groups, which was initiated in [CGLMS22]. Endowed with the Chabauty topolology, the set of subgroups $\mathrm{Sub}(\Gamma)$ of any infinite countable group Γ is a closed subset of the Cantor space $\{0,1\}^\Gamma$. A particular subset of $\mathrm{Sub}(\Gamma)$ that we are interested in is the **perfect kernel** $\mathcal{K}(\Gamma)$

Date: May 13, 2025.

of Γ, *i.e.* the largest closed subset without isolated point. This subset is invariant under Γ-conjugation. We are interested in the dynamics induced by this action. More precisely, we are interested in finding subsets of the perfect kernel on which the action is highly topologically transitive, or even topologically μ-mixing. Recall that an action of a group Γ on a topological space X is **highly topologically transitive** (HTT) if for every $r \in \mathbb{N}$ and for every non empty open subsets $U_1, ..., U_r,\ V_1, ..., V_r$, there exists $g \in \Gamma$ such that $g \cdot U_i \cap V_i \neq \emptyset$ for every $i \in [\![1, r]\!]$. The authors of [HMO24] introduced a strenghtening of this notion, called topological μ-mixing. Given a probability measure μ on a countable group Γ, an action of Γ on a topological space X is called **topologically μ-mixing** if for every non empty open subsets $U, V \subseteq X$, denoting by $(G_k)_{k\in\mathbb{N}}$ a sequence of independant μ-distributed random variables and by $(S_k)_{k\in\mathbb{N}} = (G_1 \dots G_k)_{k\in\mathbb{N}}$ the random walk on Γ with step distribution μ, one has

$$\lim_{k\to\infty} \mathbb{P}\left(G_1 \dots G_k \cdot U \cap V \neq \emptyset\right) = 1.$$

The authors of [AG23] and [HMO24] studied independently the space $\mathrm{Sub}(\Gamma)$ in the case where Γ acts on a hyperbolic space with "vanishing" stabilizers. More formally, in the case where the hyperbolic space is a tree $\mathcal{T}$ (and the action is minimal and irreducible), this simply means that the action of Γ on $\mathcal{T}$ is **acylindrical**, *i.e.* there exists $R > 0$ such that the stabilizer of any path of length larger than R is trivial. The following statements are applications of their results to this particular setting. In [AG23], the authors proved that in this case, the perfect kernel of Γ contains the closure of the set $\mathrm{Sub}_{|\bullet\backslash\mathcal{T}|\infty}(\Gamma)$ of subgroups Λ of Γ satisfying that the quotient graph $\Lambda\backslash\mathcal{T}$ is infinite:

$$\overline{\mathrm{Sub}_{|\bullet\backslash\mathcal{T}|\infty}(\Gamma)} \subseteq \mathcal{K}(\Gamma).$$

Moreover, this subset is invariant under conjugation and the action of Γ on $\overline{\mathrm{Sub}_{|\bullet\backslash\mathcal{T}|\infty}(\Gamma)}$ is HTT. The authors of [HMO24] studied a particular subset of $\mathrm{Sub}_{|\bullet\backslash\mathcal{T}|\infty}(\Gamma)$, namely the set of infinite index $\mathcal{T}$-convex cocompact subgroups $\mathrm{Sub}^{qc}_{\infty}(\Gamma \curvearrowright \mathcal{T})$. Recall that a subgroup $\Lambda \leq \Gamma$ is called **$\mathcal{T}$-convex cocompact** if it acts properly on $\mathcal{T}$ (*i.e.* with finite vertex stabilizers), with quasi-convex orbits (*i.e.* for any vertex $v \in \mathcal{T}$, there exists $\eta > 0$ such that the reduced edge path connecting two vertices of $\Lambda \cdot v$ remains at distance $< \eta$ from $\Lambda \cdot v$). On the closure of this subset, they proved that the action by conjugation is even topologically μ-mixing for every measure μ on Γ whose support is bounded, symmetric (*i.e.* stable under inversion), and generates Γ.

A Baumslag-Solitar group $\mathrm{BS}(m, n)$ is a typical example whose action on its Bass-Serre tree $\mathcal{T}_{m,n}$ is *not* acylindrical, because the stabilizer of every finite edge path is infinite cyclic. The authors of [CGLMS22] and [GLMS24] proved that this leads to a very different situation for the dynamics induced by the action by conjugation on the perfect kernel. In the case where $\min(|m|, |n|) > 1$, they proved that the perfect kernel consists exactly in the set $\mathrm{Sub}_{|\bullet\backslash\mathcal{T}_{m,n}|\infty}(\mathrm{BS}(m, n))$, and they uncovered a countably infinite invariant partition of the perfect kernel $\mathcal{K}(\mathrm{BS}(m, n)) = \bigsqcup_{N\in\mathcal{Q}_{m,n}} \mathcal{K}_N$ (where $\mathcal{Q}_{m,n}$ is an infinite subset of $\mathbb{N}^* \sqcup \{\infty\}$ that contains ∞) such that

- $\mathcal{K}_N$ is open for every finite $N \in \mathcal{Q}_{m,n}$ (and also closed iff $|m| = |n|$);
- $\mathcal{K}_\infty$ is closed;
- the action by conjugation on $\mathcal{K}_N$ is HTT for every $N \in \mathcal{Q}_{m,n}$.

Notice that the existence of disjoint invariant open subsets prevents the action on $\mathcal{K}(\mathrm{BS}(m,n))$ from being HTT. However, the last item may make us wonder if the action is also topologically μ-mixing on each piece. The following result shows that this is false in general:

Theorem 1.1. *Let $m, n \in \mathbb{Z}$ such that $\min(|m|, |n|) > 1$. Let us assume that $|m| \neq |n|$. Then, there exists a probability measure μ whose support is finite, symmetric and generates $\mathrm{BS}(m,n)$, such that for every finite $P \in \mathcal{Q}_{m,n}$, the action by conjugation of $\mathrm{BS}(m,n)$ on $\mathcal{K}_P$ is not topologically μ-mixing.*

Notice that, in the proof of Theorem 1.1, though the support of μ is symmetric, we will construct μ in such a way that $\mu(t) \neq \mu(t^{-1})$.

However, we have the following positive result:

Theorem 1.2. *Let $m, n \in \mathbb{Z}$ such that $\min(|m|, |n|) > 1$ and let $\mathcal{T}_{m,n}$ be the Bass-Serre tree of $\mathrm{BS}(m,n)$. Let μ be a probability measure on $\mathrm{BS}(m,n)$ whose support is bounded, symmetric, and generates $\mathrm{BS}(m,n)$. Then:*

1. *the action by conjugation of $\mathrm{BS}(m,n)$ on $\mathcal{K}_\infty$ is topologically μ-mixing;*
2. *if $|m| = |n|$, then for every $N \in \mathcal{Q}_{m,n}$, the action by conjugation of $\mathrm{BS}(m,n)$ on $\mathcal{K}_N$ is topologically μ-mixing.*

Notice that, for every $(m,n) \in \mathbb{Z}^2$ such that $\min(|m|, |n|) > 1$, the set $\mathcal{K}_\infty$ exactly consists of $\overline{\mathrm{Sub}^{cc}_\infty(\mathrm{BS}(m,n) \curvearrowright \mathcal{T}_{m,n})}$. Thus we extend the aforementioned result of [HMO24] to this particular case of a non acylindrical action.

Given a group Γ acting on a tree $\mathcal{T}$, the following array summarizes the results we mentioned:

	$\Gamma \curvearrowright \mathcal{T}$ acylindrical	$\mathrm{BS}(m,n) \curvearrowright \mathcal{T}_{m,n}$ $(\lvert m\rvert, \lvert n\rvert \geq 2)$ not acylindrical	
		Negative results	Positive results
HTT	$\Gamma \curvearrowright \overline{\mathrm{Sub}_{\lvert\bullet\backslash\mathcal{T}\rvert_\infty}(\Gamma)}$ HTT [AG23]	$\mathrm{BS}(m,n) \curvearrowright \mathcal{K}(\mathrm{BS}(m,n))$ $= \mathrm{Sub}_{\lvert\bullet\backslash\mathcal{T}_{m,n}\rvert_\infty}(\mathrm{BS}(m,n))$ not HTT [CGLMS22]	$\mathrm{BS}(m,n) \curvearrowright \mathcal{K}_l$ HTT $\forall l \in \mathcal{Q}_{m,n}$. [GLMS24]
μ-mixing	$G \curvearrowright \overline{\mathrm{Sub}^{cc}_{[\infty]}(\Gamma \curvearrowright \mathcal{T})}$ $\subseteq \overline{\mathrm{Sub}_{\lvert\bullet\backslash\mathcal{T}\rvert_\infty}(\Gamma)}$ μ-mixing. [HMO24]	$\mathrm{BS}(m,n) \curvearrowright \mathcal{K}_l$ not μ-mixing in general if $\lvert m\rvert \neq \lvert n\rvert$. Theorem 1.1	$\mathrm{BS}(m,\pm m) \curvearrowright \mathcal{K}_l$ μ-mixing $\forall l \in \mathcal{Q}_{m,n} \cap \mathbb{N}^*$; $\mathrm{BS}(m,n) \curvearrowright \mathcal{K}_\infty$ $= \overline{\mathrm{Sub}^{cc}_\infty(\mathrm{BS}(m,n) \curvearrowright \mathcal{T}_{m,n})}$ also μ-mixing $\forall m,n$. Theorem 1.2

The paper is organized as follows. First we recall some background around Baumslag-Solitar groups and Bass-Serre theory. We recall the main tools and the decomposition of the perfect kernel of $\mathrm{BS}(m,n)$ introduced in [CGLMS22]. Then, we build a measure μ supported on $\{b, b^{-1}, t, t^{-1}\}$ such that $\mu(t) \neq \mu(t^{-1})$ to prove Theorem 1.1, and we finally prove Theorem 1.2 using a result of Cartwright and Soardi.

Acknowledgements. I express my gratitude to my PhD advisor Damien Gaboriau for his supervision and for his valuable suggestions during the writing of this work. I also warmly thank Gregory Miermont and Mathieu Mourichoux for their enlightening explanations about random walks.

Contents

2. Preliminaries and notations

We denote by $\mathcal{P}$ the set of prime numbers. For every integer N and every prime number p, we denote by $|N|_p$ the p-adic valuation of N, that is, the largest $n \in \mathbb{N}$ such that p^n divides N. For every integers $m, n \in \mathbb{Z} \setminus \{0\}$ we denote by $m \wedge n$ their greatest common divisor, that is, the largest integer $k \in \mathbb{N}$ dividing both m and n. By convention, $\infty \wedge n = n \wedge \infty = |n|$ for every $n \in \mathbb{N}^*$. By "countable" we mean finite or in bijection with $\mathbb{N}$. For any finite set F, we denote by $|F|$ its cardinality. We use the same convention as in [Ser77] about graphs. We denote by $\mathcal{V}(\mathcal{G})$ the set of vertices of an oriented graph $\mathcal{G}$ and by $\mathcal{E}(\mathcal{G})$ its set of edges. We denote by $\mathcal{E}^+(\mathcal{G})$ (*resp.* $\mathcal{E}^-(\mathcal{G})$) its set of positive (*resp.* negative) edges, and by $\mathbf{s} : \mathcal{E}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$ and $\mathbf{t} : \mathcal{E}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$ the source and target maps, respectively. For any edge E, we denote by $\overline{E}$ the reversed edge. By the half-graph of an edge E, we mean the connected component of $\mathcal{G} \setminus \{E\}$ that contains $\mathbf{t}(E)$. If this half-graph is a tree, we call it a half-tree.

2.1. **Space of subgroups.** Given an infinite countable group Γ, one can endow its set of subgroups $\mathrm{Sub}(\Gamma)$ with the **Chabauty topology**, which comes from the natural inclusion $\mathrm{Sub}(\Gamma) \hookrightarrow \{0,1\}^{\Gamma}$ of $\mathrm{Sub}(\Gamma)$ into the Cantor set. A basis of neighborhoods is given by the following family of clopen sets

$$\mathcal{V}(O,I) = \{\Lambda \leq \Gamma \mid I \subseteq \Lambda \text{ and } \Lambda \cap O = \emptyset\}$$

(where I and O are finite subsets of Γ).

One has a correspondence between subgroups of Γ and isomorphism classes of transitive right actions of Γ on pointed countable sets, which yields a correspondence between conjugacy classes of subgroups of Γ and isomorphism classes of transitive right actions of Γ on countable sets, where the action by conjugation amounts to changing the base point. It is given by the bijection

$$\begin{array}{ccc} \{\text{isomorphism classes of pointed transitive right } \Gamma\text{-actions}\} & \to & \mathrm{Sub}(\Gamma) \\ (X, x_0) \curvearrowleft^{\alpha} \Gamma & \mapsto & \mathrm{Stab}_{\alpha}(x_0) \end{array} \tag{2}$$

whose inverse is given by

$$\begin{array}{ccc} \mathrm{Sub}(\Gamma) & \to & \{\text{isomorphism classes of pointed transitive right } \Gamma\text{-actions}\} \\ \Lambda & \mapsto & (\Lambda\backslash\Gamma, \Lambda) \curvearrowleft \Gamma \end{array}.$$

The Chabauty topology can be defined on the set of pointed transitive right Γ-actions thanks to **Schreier graphs**. Given a symmetric generating set S of Γ and a right Γ-action α on a pointed countable set (X, x_0), one can define the (rooted) Schreier graph of α as follows: its set of vertices is X and, for every $x \in X$ and $s \in S$, there is a positive edge labeled s with source x and target $x \cdot s$, whose opposite edge is labeled s^{-1} (and has source $x \cdot s$ and target x). Its root is the point x_0. The set of pointed transitive right Γ-actions can be endowed with the following topology. A basis of neighborhoods of a pointed transitive right action $(X, v) \curvearrowleft^{\alpha} \Gamma$ is given by the set of actions whose Schreier graph has the same R-ball around the origin (for $R > 0$):

$$V_R = \{(X', v') \curvearrowleft^{\beta} \Gamma, \mathrm{Sch}(\beta, v') \simeq_R \mathrm{Sch}(\alpha, v)\}.$$

Via the aforementioned identification between actions and subgroups, this is exactly the Chabauty topology. We refer to [Bon24, Section 2] for more details. From now on, we will freely identify the set of subgroups of Γ with the set of pointed transitive right actions of Γ.

By a theorem of Cantor-Bendixson, there exists a unique decomposition

$$\mathrm{Sub}(\Gamma) = \mathcal{K}(\Gamma) \sqcup C$$

where C is countable and $\mathcal{K}(\Gamma)$ is a closed subset without isolated point, called the **perfect kernel** of Γ. This is the largest closed subset of $\mathrm{Sub}(\Gamma)$ without isolated points, or equivalently, the set of subgroups all of whose neighborhoods are uncountable. See [Kec95, Section 6] for more details.

2.2. **Baumslag-Solitar groups; preactions and (m,n)-(Schreier) graphs.** In this section, we provide a quick reminder on Bass-Serre theory and we recall the main tools that were introduced in [CGLMS22] to study $\mathrm{Sub}(\mathrm{BS}(m,n))$. Let $m,n \in \mathbb{Z}$ such that $\min(|m|,|n|) > 1$. The Baumslag-Solitar group of parameters (m,n) is the group Γ defined by the presentation (1).

2.2.1. *Some background on Bass-Serre theory.* As an HNN-extension, the group $\mathrm{BS}(m,n)$ acts on its **Bass-Serre tree** $\mathcal{T}_{m,n}$ (on the left): this is the infinite oriented tree all of whose vertices have m incoming edges and n outgoing edges. This tree arises as follows: it is obtained from the (right) Cayley graph of $\mathrm{BS}(m,n)$ (with respect to the generating set $\{b, b^{-1}, t, t^{-1}\}$) by shrinking all the $\langle b \rangle$-orbits. There is a natural map $p : \mathrm{Cay}(\mathrm{BS}(m,n)) \to \mathcal{T}_{m,n}$ (applying [CGLMS22, Definition 3.10] to the free action of $\mathrm{BS}(m,n)$ on itself); it sends

- the vertex γ of the Cayley graph to the vertex $\gamma\langle b \rangle$ of the Bass-Serre tree;
- the positive edge $(\gamma, \gamma t)$ of the Cayley graph to the positive edge $\gamma\langle b^n \rangle$ of the Bass-Serre tree;
- the negative edge $(\gamma, \gamma t^{-1})$ of the Cayley graph to the negative edge $\gamma t^{-1}\langle b^m \rangle$ of the Bass-Serre tree;
- the edges of the form $(\gamma, \gamma b^k)$ to the single vertex $\gamma\langle b \rangle$.

The action of $\mathrm{BS}(m,n)$ on the set of (positive) edges of $\mathcal{T}_{m,n}$ is defined by: $\gamma \cdot g\langle b^n \rangle = \gamma g \langle b^n \rangle$. The action of $\mathrm{BS}(m,n)$ on its Bass-Serre tree has a single orbit of vertices and the quotient graph $\mathrm{BS}(m,n)\backslash\mathcal{T}_{m,n}$ is a loop. The vertex $p(1) = \langle b \rangle$ is the root of the Bass-Serre tree. See [CGLMS22, Section 2.3, Section 3] for more details.

2.2.2. *Normal forms.* A **word** of $\mathrm{BS}(m,n)$ is a sequence $s = (u_k)_{k\in\mathbb{N}}$ of elements of $\mathrm{BS}(m,n)$ such that there exists $k_0 \in \mathbb{N}$ such that

- for every $k < k_0$, one has $u_k \in \{b, b^{-1}, t, t^{-1}\}$;
- $u_k = 1$ for every $k \geq k_0$.

The element associated to s in $\mathrm{BS}(m,n)$ is the product $\mathfrak{s} = u_1 \dots u_{k_0}$. A **subword** s' of s is a sequence of the form $s' = (u_1, \dots, u_{l_0}, 1, \dots)$, for some $l_0 \leq k_0$. By an abuse of notation, we will freely identify the word s and the element $\mathfrak{s}$ of $\mathrm{BS}(m,n)$ (keeping in mind that the writing of $\mathfrak{s}$ as a product of elements of $\{b, b^{-1}, t, t^{-1}\}$ is not unique!). In particular, the subwords of an element of $\mathrm{BS}(m,n)$ strongly depends on the chosen writing of this element as a product of elements in $\{b, b^{-1}, t, t^{-1}\}$.

Example 2.1. For instance, the subwords of the word $tbbt^{-1}b^{-1}b^{-1}b^{-1}$ in $\mathrm{BS}(2,3)$ are $\emptyset$, t, tb, tbb, $tbbt^{-1}$, $tbbt^{-1}b^{-1}$, $tbbt^{-1}b^{-1}b^{-1}$, $tbbt^{-1}b^{-1}b^{-1}b^{-1}$ (though this is the trivial element of $\mathrm{BS}(2,3)$).

As a particular case of [Ser77, Chapter 1, Section 5.2] (which yields the normal form for HNN-extensions), any element g of $\mathrm{BS}(m,n)$ can be uniquely written as

$$g = b^{n_1} t^{\varepsilon_1} b^{n_2} \dots b^{n_r} t^{\varepsilon_r} b^{n_{r+1}}$$

where $n_i \in \mathbb{Z}$ for every $i \in [\![1, r+1]\!]$, and $\varepsilon_i \in \{1, -1\}$ for every $i \in [\![1, r]\!]$, and

- if $\varepsilon_i = 1$, then $n_{i+1} \in [\![1, |m| - 1]\!]$;
- if $\varepsilon_i = -1$, then $n_{i+1} \in [\![1, |n| - 1]\!]$.

This is the **normal form** of g. If g is written in its normal form, we say that g is **reduced**. The **height** of an element $g = b^{n_1} t^{\varepsilon_1} b^{n_2} \dots b^{n_r} t^{\varepsilon_r} b^{n_{r+1}} \in \mathrm{BS}(m, n)$ written in its normal form is the integer

$$\mathfrak{h}(g) = r.$$

Remark 2.2. This is also the distance

$$\mathfrak{h}(g) = d_{\mathcal{T}_{m,n}}(v, g \cdot v),$$

where $v = \langle b \rangle$ is the root of the Bass-Serre tree $\mathcal{T}_{m,n}$ of $\mathrm{BS}(m, n)$.

2.2.3. *Preactions and (m, n)-graphs.* Any subgroup $\Lambda \leq \mathrm{BS}(m, n)$ acts on $\mathcal{T}_{m,n}$. As the following diagram

$$\begin{tikzcd}
 & \mathrm{Cay}(\mathrm{BS}(m,n)) \arrow[dl] \arrow[dr] & \\
\mathrm{Sch}(\Lambda) = \Lambda\backslash\mathrm{Cay}(\mathrm{BS}(m,n)) \arrow[dr] & & \mathrm{Cay}(\mathrm{BS}(m,n))/\langle b\rangle = \mathcal{T}_{m,n} \arrow[dl] \\
 & \Lambda\backslash\mathrm{Cay}(\mathrm{BS}(m,n))/\langle b\rangle &
\end{tikzcd}$$

commutes, the quotient graph $\Lambda\backslash\mathcal{T}_{m,n}$ can be obtained by shrinking all the $\langle b\rangle$-orbits of the Schreier graph of Λ (with respect to the generating set $\{b, b^{-1}, t, t^{-1}\}$).

Motivated by this observation, we define a (right) **preaction** on a pointed countable set (X, x_0) as a couple of partial bijections (β, τ) such that β is a genuine bijection of X, $\mathrm{dom}(\tau)$ is β^n-invariant, $\mathrm{rng}(\tau)$ is β^m-invariant and $x \cdot \tau\beta^m = x \cdot \beta^n\tau$. The **Schreier (m, n)-graph** $\mathrm{Sch}(\alpha)$ of a preaction α on a countable set X is the oriented graph whose set of vertices is X and whose (positive) edges are either of the form $(x, x\cdot\beta)$, or of the form $(x, x\cdot\tau)$ (for $x \in X$). Every path c in $\pi_1(\mathrm{Sch}(\alpha), x_0)$ is labeled by a word whose letters lie in $\{b, b^{-1}, t, t^{-1}\}$, thus defines an element $\psi(c) \in \Gamma$. The map $\psi : \pi_1(\mathrm{Sch}(\alpha), x_0) \to \Gamma$ is a group morphism, and the image of this map is called the **stabilizer** of the point x_0 for the preaction α, and denoted by $\mathbf{Stab}_{\boldsymbol{\alpha}}\boldsymbol{(x_0)}$. The **$(m, n)$-graph** $\mathcal{G}_\alpha$ of a preaction α is the Schreier (m, n)-graph of α all of whose β-orbits have been shrunk and labeled by their cardinalities. More specifically

- its set of vertices is $X/\langle\beta\rangle$ and every vertex $x\langle\beta\rangle$ is labeled by the cardinality $|x \cdot \langle\beta\rangle|$;
- its set of positive (*resp.* negative) edges is $\mathrm{dom}(\tau)/\langle\beta^n\rangle$ (*resp.* $\mathrm{dom}(\tau)/\langle\beta^m\rangle$);
- the target and source maps are defined by $\mathbf{s}(x \cdot \langle\beta^n\rangle) = x \cdot \langle\beta\rangle$ and $\mathbf{t}(x \cdot \langle\beta^n\rangle) = x\tau \cdot \langle\beta\rangle$. Moreover, $\overline{x \cdot \langle\beta^n\rangle} = x\tau \cdot \langle\beta^m\rangle$.

A preaction $\alpha = (\beta, \tau)$ is called **transitive** if its (m, n)-graph is connected. It is called **saturated** if $\mathrm{dom}(\beta) = \mathrm{rng}(\beta) = X$. If α is a saturated and transitive preaction on a pointed countable set (X, x_0), the data of the (m,n)-graph of α is equivalent to the data of the graph of groups of $\mathrm{Stab}_\alpha(x_0)$, given by its action on the Bass-Serre tree $\mathcal{T}_{m,n}$. See [CGLMS22, Section 3] for more details.

Remark 2.3. Assume that $\Lambda \leq \Gamma$ is a finitely generated subgroup whose graph of groups (induced by its action on $\mathcal{T}_{m,n}$) is infinite (that is to say, $\Lambda \backslash \mathcal{T}_{m,n}$ is infinite). Let us write this graph of groups as an increasing union of finite subgraphs K_n. Then, denoting by Λ_n the fundamental group of the graph of groups K_n, there exists $n_0 \in \mathbb{N}$ such that $\Lambda_n = \Lambda_{n_0}$ for every $n \geq n_0$. In particular, the preimage $\pi^{-1}(K_{n_0})$ is a proper invariant subtree of $\mathcal{T}_{m,n}$.

An abstract $\boldsymbol{(m,n)}$**-graph** is then an oriented labeled graph all of whose vertices are labeled by an integer or ∞ and that satisfies the following arithmetical properties:

(1) every vertex labeled N has at most $N \wedge n$ outgoing edges and at most $N \wedge m$ incoming edges;
(2) **(Transfer Equation)** for every positive edge with source labeled N and target labeled M, one has

$$\frac{N}{N \wedge n} = \frac{M}{M \wedge m}. \tag{3}$$

It is called **saturated** if equalities hold for every vertex in the first item.

We will make use of the following lemma that gives the existence of a particular extension of a given preaction. A proof can be found in [GLMS24, Theorem 4.6]:

Lemma 2.4 (Maximal forest saturation)**.** *For any transitive and non saturated preaction α on a pointed countable set (X, x_0), there exists a unique transitive action β (up to isomorphism) such that:*

- *β extends α (in particular, $\mathcal{G}_\beta$ contains $\mathcal{G}_\alpha$ as a subgraph);*
- $\mathrm{Stab}_\beta(x_0) = \mathrm{Stab}_\alpha(x_0)$

Moreover, denoting by $\mathcal{G}_\beta$ the (m,n)-graph of β, this action has the following properties:

- *the subgraph induced by the set of vertices of $\mathcal{G}_\beta \smallsetminus \mathcal{G}_\alpha$ is an infinite forest $\mathcal{F}$;*
- *the subgraph induced by the set of vertices of $\mathcal{G}_\alpha$ is $\mathcal{G}_\alpha$;*
- *for any edge $E \in \mathcal{E}(\mathcal{G}_\beta) \smallsetminus \mathcal{E}(\mathcal{G}_\alpha)$, denoting by N the label of $\mathbf{s}(E)$ and by M the label of $\mathbf{t}(E)$:*
  - *if the half-tree of E is in $\mathcal{F}$, then $M = \frac{N|m|}{N \wedge n}$;*
  - *otherwise, i.e. if the half-tree of $\overline{E}$ is in $\mathcal{F}$, then $N = \frac{M|n|}{M \wedge m}$.*

The Γ-action β constructed in Lemma 2.4 is called the **maximal forest saturation action of** $\boldsymbol{\alpha}$**.**

Remark 2.5. If Λ is a finitely generated subgroup of Γ, then there exists a preaction α on a pointed countable set (X, x_0) whose (m,n)-graph is finite and such that $\mathrm{Stab}_\alpha(x_0) = \Lambda$. The Γ-right action associated to Λ by the correspondence (2) is exactly the maximal forest saturation action β of α given by Lemma 2.4.

Remark 2.6. We keep the notations of Lemma 2.4. Identifying (m, n)-graphs with graphs of subgroups, Remark 2.3 tells us that, denoting by $\pi : \mathcal{T}_{m,n} \to \mathcal{G}_\beta$ the projection, the preimage $\pi^{-1}(\mathcal{G}_\alpha)$ is a proper $\mathrm{Stab}_\alpha(x_0)$-invariant subtree of $\mathcal{T}_{m,n}$.

Keeping the notations of Lemma 2.4, in the case where $|m| = |n|$ or when the $\langle b\rangle$-orbits of α are infinite, the forest $\mathcal{F}$ is a collection of half-subtrees of $\mathcal{T}_{m,n}$. More formally:

Proposition 2.7. *Let α be a transitive and non saturated preaction, and let β be its maximal forest saturation action. Let K be the (m, n)-graph of α and let $\mathcal{G}$ be the (m, n)-graph of β. Let $\mathcal{F}$ be the forest induced by the set of vertices of $\mathcal{G} \smallsetminus K$. Let us assume that*

- *either $|m| = |n|$;*
- *or the $\langle b\rangle$-orbits of α are infinite.*

Let E be any edge with source in K and target outside K and let $\widehat{\mathcal{T}} \subseteq \mathcal{F}$ be the half-tree of E. Then, the projection $\pi : \mathcal{T}_{m,n} \to \mathcal{G}$ induces a homeomorphism between any connected component of $\pi^{-1}\left(\widehat{\mathcal{T}}\right)$ and $\widehat{\mathcal{T}}$.

Proof. By the properties of the maximal forest saturation action (*cf.* Lemma 2.4), any vertex of $\mathcal{G} \smallsetminus K$ has $|m|$ incoming edges and $|n|$ outgoing edges if $|m| = |n|$ or the labels of the vertices of $\mathcal{G}$ are infinite (*i.e.* the $\langle b\rangle$-orbits of α are infinite). As any vertex of $\mathcal{T}_{m,n}$ also has $|m|$ incoming edges and $|n|$ outgoing edges, this tells us that π induces a locally injective graph morphism between any connected component of $\pi^{-1}\left(\widehat{\mathcal{T}}\right)$ and $\widehat{\mathcal{T}}$. Finally, as $\widehat{\mathcal{T}}$ is a tree, [Ser77, Section 4.5, Lemma 5] tells us that this induced graph morphism is in fact a homeomorphism. $\square$

Given a preaction α on a pointed countable set (X, x_0), one has a projection $p_\alpha : \mathrm{Sch}(\alpha) \to \mathcal{G}_\alpha$,

- that shrinks the $\langle\beta\rangle$-orbits;
- that sends the edge labeled t connecting x to $x \cdot \tau$ to the edge $x \cdot \langle\beta^n\rangle$ for every $x \in X$

(*cf.* [CGLMS22, Definition 3.10] for more details).

Any couple (γ, x) where $\gamma = b^{n_1}t^{\varepsilon_1}b^{n_2}\dots b^{n_r}t^{\varepsilon_r}b^{n_{r+1}}$ is a word and x is a vertex of $\mathrm{Sch}(\alpha)$ leads to a unique edge path $E_1, \dots, E_r$ in $\mathcal{G}_\alpha$ (whose orientation is given by the sequence of signs $(\varepsilon_1, \dots, \varepsilon_r)$ and such that $\mathbf{s}(E_i) = p_\alpha(x \cdot b^{n_1}t^{\varepsilon_1}\dots t^{\varepsilon_{i-1}}b^{n_i})$ for every $i \in [\![1, r]\!]$, and $\mathbf{t}(E_r) = p_\alpha(x \cdot \gamma))$. We say that the edge path $E_1, ..., E_r$ **derives from** x and the word γ. This observation leads to the following estimate: for any $x \in \mathcal{V}(\mathrm{Sch}(\alpha))$, one has

$$d_{\mathcal{G}_\alpha}(p_\alpha(x), p_\alpha(x \cdot \gamma)) \leq \mathfrak{h}(\gamma). \tag{4}$$

If the edge path $E_1, \dots, E_r$ is reduced, then γ is necessarily reduced. Conversely, if $\gamma = b^{n_1}t^{\varepsilon_1}b^{n_2}\dots b^{n_r}t^{\varepsilon_r}b^{n_{r+1}}$ is reduced and the edge path $E_1, \dots, E_r$ defined by (γ, x) satisfies the following conditions:

- m divides $|x \cdot b^{n_1}t^{\varepsilon_1}b^{n_2}\dots b^{n_i}t^{\varepsilon_i}\langle b\rangle|$ for every i such that $\varepsilon_i = 1$ and $\varepsilon_{i+1} = -1$;
- n divides $|x \cdot b^{n_1}t^{\varepsilon_1}b^{n_2}\dots b^{n_i}t^{\varepsilon_i}\langle b\rangle|$ for every i such that $\varepsilon_i = -1$ and $\varepsilon_i = 1$,

then the edge path $E_1, \dots, E_r$ is reduced (*cf.* [Bon24, Section 3.1, Lemma 5.9] for more details).

From now on, we will freely identify subgroups of $\mathrm{BS}(m,n)$ and pointed transitive saturated preactions.

One defines a topology on the set of transitive saturated preactions as follows: a basis of neighborhoods of a transitive saturated preaction α is given by

$$\{\beta \text{ transitive saturated preaction on a pointed countable set } (X', x') \mid$$
$$p_\beta^{-1}(B_{\mathcal{G}_\beta}(p(x'), R)) \text{ is isomorphic to } p_\alpha^{-1}(B_{\mathcal{G}_\alpha}(p(x), R)) \text{ (as labeled graphs)}\}$$

(for all $R > 0$).

Remark 2.8. Notice that the topology on the set of transitive saturated preactions is finer than the aforementioned topology on the set of Γ right-actions. In other words, if two subgroups Λ_1 and Λ_2 have close (m,n)-rooted Schreier graphs, then they are close for the Chabauty topology.

2.3. **Random walks on groups.** The definitions of this section come from [HMO24]. Let Γ be a countable group. Let $\mu : \Gamma \to [0,1]$ be a probability measure on Γ. A **random walk on Γ with step distribution $\boldsymbol{\mu}$** is a random sequence $(S_k)_{k\in\mathbb{N}} = (G_1 \dots G_k)_{k\in\mathbb{N}}$ of Γ where the G_i's are independent μ distributed random variables.

Let $(S_k)_{k\in\mathbb{N}}$ be a random walk on a group Γ with step distribution μ. A Γ-action on a Polish space X is said **topologically $\boldsymbol{\mu}$-mixing** if for any non empty open sets $U, V \subseteq X$, the following holds:

$$\lim_{k\to\infty} \mathbb{P}(S_k \cdot U \cap V \neq \emptyset) = 1.$$

Notice that any topologically μ-mixing action is also **l-topologically transitive** for every $l \in \mathbb{N}$: for every non empty open sets $U_1, \dots, U_l, V_1, \dots, V_l \subseteq X$, there exists an element $g \in \Gamma$ such that

$$g \cdot U_i \cap V_i \neq \emptyset, \ \forall i \in [\![1, l]\!]$$

(see [HMO24, Proposition 1.2]).

Given a probability measure $\mu : \Gamma \to [0,1]$, we denote by $\mathrm{Supp}(\mu)$ its **support**, that is to say, the set

$$\mathrm{Supp}(\mu) = \{g \in \Gamma \mid \mu(g) > 0\}.$$

We say that the support of μ is **generating** if $\mathrm{Supp}(\mu)$ generates Γ as a semi-group, and that it is **symmetric** if $\mathrm{Supp}(\mu)$ is stable under inversion.

If Γ acts isometrically on a metric space X, we say that the support of μ is **bounded** (with respect to this action) if $\mathrm{Supp}(\mu)(x)$ is a bounded subset of X for some (equivalently for all) $x \in X$.

Remark 2.9. Let $\Gamma = \mathrm{BS}(m,n)$, that acts on its Bass-Serre tree $\mathcal{T}_{m,n}$ and let μ be a probability measure on Γ. Then, the support of μ is bounded iff the set of integers

$$\{\mathfrak{h}(g), g \in \mathrm{Supp}(\mu)\}$$

is bounded. We draw the attention of the reader to the fact that in this case, the support of μ need not be finite.

Throughout the paper, we will use capital letters to refer to random variables.

2.4. **Phenotype of a subgroup of** $\mathrm{BS}(m,n)$**.** In this section, we recall the computation of the perfect kernel and the construction of the decomposition of $\mathcal{K}(\mathrm{BS}(m,n))$ obtained in [CGLMS22]. The authors proved the following result:

Theorem 2.10. *Let $(m,n) \in (\mathbb{Z} \smallsetminus \{0\})^2$ such that $|m| \neq 1$ and $|n| \neq 1$. Then*

$$\mathcal{K}(\mathrm{BS}(m,n)) = \{\Lambda \leq \mathrm{BS}(m,n) \mid \quad \text{the } (m,n)\text{-graph of } \Lambda \text{ is infinite}\}.$$

To understand the dynamics induced by the action by conjugation, the authors introduced the notion of **phenotype**. This is a function $\mathrm{Ph}_{m,n} : \mathbb{N} \sqcup \{\infty\} \to \mathbb{N} \sqcup \{\infty\}$ which is constant on the set of labels of any connected (m,n)-graph. It is defined by

$$\mathrm{Ph}_{m,n}(N) := \begin{cases} \prod_{p \in \mathcal{P}, |m|_p = |n|_p \text{ and } |N|_p > |n|_p} p^{|N|_p} & \text{if } N \in \mathbb{N} \\ \infty & \text{if } N = \infty. \end{cases} \tag{5}$$

The phenotype of a connected (m,n)-graph $\mathcal{G}$ is defined as follows: denoting by N any label of $\mathcal{G}$, then $\mathbf{Ph}_{m,n}(\mathcal{G}) = \mathrm{Ph}_{m,n}(N)$.

The phenotype $\mathbf{Ph}_{m,n}(\Lambda)$ of a subgroup $\Lambda \leq \mathrm{BS}(m,n)$ is then the phenotype of its (m,n)-graph. As the labels of the (m,n)-graph of Λ encode the intersections of the conjugates of Λ with $\langle b \rangle$, one also has $\mathbf{Ph}_{m,n}(\Lambda) = \mathrm{Ph}_{m,n}([\langle b \rangle : \Lambda \cap \langle b \rangle])$ (as defined in (5)).

The authors of [CGLMS22] proved that this quantity is invariant under conjugation and that the set $\mathcal{Q}_{m,n} = \mathbf{Ph}_{m,n}(\mathrm{Sub}(\Gamma))$ is infinite. Thus, we get an infinite countable invariant partition of $\mathrm{Sub}(\mathrm{BS}(m,n)) = \bigsqcup_{N \in \mathcal{Q}_{m,n}} \mathbf{Ph}_{m,n}^{-1}(N)$. Moreover, they proved the following result:

Theorem 2.11. *Let $(m,n) \in (\mathbb{Z} \smallsetminus \{0\})^2$ such that $|m| \neq 1$ and $|n| \neq 1$. Then, for any $N \in \mathcal{Q}_{m,n}$:*

- *$\mathcal{K}(\mathrm{BS}(m,n)) \cap \mathbf{Ph}_{m,n}^{-1}(N)$ is non empty and the action by conjugation of $\mathrm{BS}(m,n)$ on it is topologically transitive.;*
- *if $N \in \mathbb{N}$, the piece $\mathbf{Ph}_{m,n}^{-1}(N)$ is open;*
- *it is also closed if and only if $|m| = |n|$;*
- *the piece $\mathbf{Ph}_{m,n}^{-1}(\infty)$ is closed.*

In [GLMS24], the same authors even proved that the action of $\mathrm{BS}(m,n)$ by conjugation on $\mathcal{K}(\mathrm{BS}(m,n)) \cap \mathbf{Ph}_{m,n}^{-1}(N)$ is highly topologically transitive if $\mathbf{Ph}_{m,n}^{-1}(N) \neq \emptyset$.

Remark 2.12. In the case where $|m| = |n|$, it is not hard to check that the phenotype of a non zero integer N satisfies the following equality:

$$\mathrm{Ph}_{\pm n,n}(N) \left(\prod_{p \in \mathcal{P}, \left|\mathrm{Ph}_{\pm n,n}(N)\right|_p = 0} p^{|n|_p} \right) = \frac{Nn}{N \wedge n}.$$

In particular, if α is a transitive and non saturated preaction, denoting by β its maximal forest saturation action, any vertex of $\mathcal{G}_\beta \smallsetminus \mathcal{G}_\alpha$ is labeled $\mathbf{Ph}_{\pm n,n}(\mathcal{G}_\alpha)\left(\prod_{p\in\mathcal{P},|\mathbf{Ph}_{\pm n,n}(\mathcal{G}_\alpha)|_p=0} p^{|n|_p}\right)$.

3. Random walks on the Bass-Serre tree and on (m,n)-graphs

Let $m,n\in\mathbb{Z}$ such that $\min(|m|,|n|)>1$. Let $\mathcal{T}_{m,n}$ be the Bass-Serre tree of the Baumslag-Solitar group $\Gamma := \mathrm{BS}(m,n)$ of parameters (m,n).

3.1. Proof of Theorem 1.1.

In this section, we assume that $|m|\neq|n|$. The goal is to build a probability measure μ whose support is $\{b,b^{-1},t,t^{-1}\}$ such that, for every finite $P\in\mathcal{Q}_{m,n}$, the action by conjugation on $\mathbf{Ph}_{m,n}^{-1}(P)$ is not topologically μ-mixing.

To prove Theorem 1.1, we will consider a probability measure supported on $\{b,b^{-1},t,t^{-1}\}$ satisfying $\mu(t)\neq\mu\left(t^{-1}\right)$. We will use the following deterministic result:

Proposition 3.1. *Let $g_1,\dots,g_r\in\{b,b^{-1},t,t^{-1}\}$. For every $i\in[\![1,r]\!]$, let*

$$\mathfrak{h}_i^+ := |\{j\in[\![1,i]\!] \mid g_j=t\}|$$

and

$$\mathfrak{h}_i^- = \left|\{j\in[\![1,i]\!] \mid g_j=t^{-1}\}\right|$$

(and $\mathfrak{h}_0^+=\mathfrak{h}_0^-=0$). Let α be a preaction on a pointed countable set (X,x) and p be a prime number such that

- *$|m|_p>|n|_p$;*
- *b and g_1 are defined on x, and for every $i\in[\![2,r]\!]$, the elements b and g_i are defined on $x\cdot g_1\cdot\ldots\cdot g_{i-1}$;*
- *the cardinal N of the $\langle b\rangle$-orbit of x satisfies $|N|_p>|m|_p$.*

Then, for every $i\in[\![0,r]\!]$, the cardinality N_i of the $\langle b\rangle$-orbit of $x\cdot g_1\ldots g_i$ satisfies

$$|N_i|_p = (\mathfrak{h}_i^+-\mathfrak{h}_i^-)(|m|_p-|n|_p)+|N|_p$$

(where $N_0=N$ and $x\cdot g_1\ldots g_i=x$ if $i=0$).

Proof. We proceed by induction on $i\in[\![0,r]\!]$.
Base case. The result is clear for $i=0$.
Induction step. Let us assume that $i\in[\![0,r-1]\!]$ and that

$$|N_i|_p = (\mathfrak{h}_i^+-\mathfrak{h}_i^-)(|m|_p-|n|_p)+|N|_p.$$

We distinguish three cases:
1st case: $g_{i+1}\in\{b,b^{-1}\}$. In this case, the $\langle b\rangle$-orbits of $x\cdot g_1\ldots g_i$ and $x\cdot g_1\ldots g_{i+1}$ are the same. Thus, $N_{i+1}=N_i$. Moreover, $\mathfrak{h}_{i+1}^+=\mathfrak{h}_i^+$ and $\mathfrak{h}_{i+1}^-=\mathfrak{h}_i^-$, thus $|N_{i+1}|_p=(\mathfrak{h}_{i+1}^+-\mathfrak{h}_{i+1}^-)(|m|_p-|n|_p)+|N|_p$.

2nd case: $g_{i+1} = t$**.** In this case, one has $\mathfrak{h}_{i+1}^+ = \mathfrak{h}_i^+ + 1$ and $\mathfrak{h}_{i+1}^- = \mathfrak{h}_i^-$. By the Transfer Equation (3), one has $\frac{N_i}{N_i \wedge n} = \frac{N_{i+1}}{N_{i+1} \wedge m}$, which implies that

$$|N_i|_p - \min(|N_i|_p, |n|_p) = |N_{i+1}|_p - \min(|N_{i+1}|_p, |m|_p). \tag{6}$$

By the induction hypothesis

$$\begin{aligned} |N_i|_p &= (\mathfrak{h}_i^+ - \mathfrak{h}_i^-)(|m|_p - |n|_p) + |N|_p \\ &\geq |N|_p \text{ by the assumption made on } \mathfrak{h}_i^+, \mathfrak{h}_i^- \text{ and the assumption made on } p \\ &> |m|_p \text{ by the assumption made on } N \\ &> |n|_p \text{ by the assumption made on } p. \end{aligned}$$

Thus by Equation (6):

$$\begin{aligned} |N_{i+1}|_p - \min(|N_{i+1}|_p, |m|_p) &= |N_i|_p - |n|_p \\ &> 0 \end{aligned}$$

so necessarily

$$\begin{aligned} |N_{i+1}|_p &= |N_i|_p + |m|_p - |n|_p \\ &= (\mathfrak{h}_i^+ - \mathfrak{h}_i^- + 1)(|m|_p - |n|_p) + |N|_p \\ &= (\mathfrak{h}_{i+1}^+ - \mathfrak{h}_{i+1}^-)(|m|_p - |n|_p) + |N|_p, \end{aligned}$$

which concludes the induction step in the case where $g_{i+1} = t$.
3rd case: $g_{i+1} = t^{-1}$**.** In particular, $\frac{N_i}{N_i \wedge m} = \frac{N_{i+1}}{N_{i+1} \wedge n}$ and $\left(\mathfrak{h}_{i+1}^+, \mathfrak{h}_{i+1}^-\right) = \left(\mathfrak{h}_i^+, \mathfrak{h}_i^- + 1\right)$. This last case is very similar, so we leave it to the reader. □

Before proving Theorem 1.1, let us recall some basic facts about one dimensional random walks.

Proposition 3.2. *Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of iid random variables, valued in $\{-1, 0, 1\}$, such that $\mathbb{P}(X_i = 1) > \mathbb{P}(X_i = -1)$. Let us denote by $Z_n := \sum_{i=1}^n X_i$ (where $Z_0 = 0$), and by $p_+ := \mathbb{P}(X_i = 1)$ and $p_- := \mathbb{P}(X_i = -1)$. Then*
(1) $\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n} Z_n = p_+ - p_-\right) = 1$;
(2) $\mathbb{P}(Z_n > 0,\ \forall n > 0) = p_+ - p_-$.

Proof. The first item results from the strong law of large numbers.

For the second one, notice that, denoting by $p_{k,l}$ the probability that Z_n reaches l, starting from $k > l$, one has

$$p_{1,0} = \mathbb{P}(X_1 = -1) + \mathbb{P}(X_1 = 0)p_{1,0} + \mathbb{P}(X_1 = 1)p_{2,0}$$

As $p_{2,0} = p_{2,1}p_{1,0}$ and $p_{k,l} = p_{k-l,0}$ for any $k > l$, one deduces that $p = p_{1,0}$ satisfies the following equation:

$$p_+p^2 - (p_+ + p_-)p + p_- = 0$$

which leads to $p = \frac{p_-}{p_+}$ or $p = 1$. Thus

$$\begin{aligned}\mathbb{P}(Z_n > 0,\ \forall n > 0) &= \mathbb{P}(X_1 = 1)(1 - p_{1,0}) \\ &= \begin{cases} p_+ - p_- & \text{if } p_{1,0} = \frac{p_-}{p_+} \\ 0 & \text{if } p_{1,0} = 1. \end{cases}\end{aligned} \tag{7}$$

We want to show that $\mathbb{P}(Z_n > 0,\ \forall n > 0) = p_+ - p_-$. The first item of the proposition implies that

$$\mathbb{P}(\exists n_0 \in \mathbb{N}, Z_{n_0} = 0 \ \text{ and } \ Z_n > 0, \forall n > n_0) = 1.$$

In particular, there exists $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}(Z_{n_0} = 0 \ \text{ and } \ Z_n > 0, \forall n > n_0) > 0.$$

In particular,

$$\begin{aligned}\mathbb{P}(Z_n > 0, \forall n > 0) &= \mathbb{P}\left(Z_n > 0, \forall n > n_0 \mid Z_{n_0} = 0\right) \\ &= \frac{\mathbb{P}(Z_{n_0} = 0 \ \text{ and } \ Z_n > 0, \forall n > n_0)}{\mathbb{P}(Z_{n_0} = 0)} \\ &> 0.\end{aligned}$$

Thus, Equation (7) implies that

$$\mathbb{P}(Z_n > 0,\ \forall n > 0) = p_+ - p_-.$$

□

We are now ready to prove Theorem 1.1, which is a particular case of the following theorem:

Theorem 3.3. *Let us assume that $|m| \neq |n|$. Let p be a prime number such that $|m|_p \neq |n|_p$. Let $\mu : \Gamma \to [0, 1]$ be a probability measure that satisfies the following properties:*

- $\mathrm{Supp}(\mu) = \{b, b^{-1}, t, t^{-1}\}$;
- $\mu(t) > \mu(t^{-1})$ *if* $|m|_p > |n|_p$;
- $\mu(t) < \mu(t^{-1})$ *if* $|m|_p < |n|_p$.

Let $P \in \mathcal{Q}_{m,n} \cap \mathbb{N}$. Then, the action by conjugation on $\mathbf{Ph}^{-1}_{m,n}(P)$ is not topologically μ-mixing.

Proof. Up to exchanging m and n, let us assume that $|m|_p > |n|_p$. Let $N \in \mathrm{Ph}^{-1}_{m,n}(P)$ such that $|N|_p > |m|_p$. Such N exists, because $\mathrm{Ph}_{m,n}(Np^r) = \mathrm{Ph}_{m,n}(N)$ for every $r \in \mathbb{N}$. Let $S_k = G_1 \dots G_k$ be the random walk with step distribution μ. Let us define a sequence of random variables $(X_i)_{i \in \mathbb{N}}$, valued in $\{-1, 0, 1\}$, as follows:

$$X_i = \begin{cases} 1 & \text{if } \ G_i = t; \\ -1 & \text{if } \ G_i = t^{-1}; \\ 0 & \text{otherwise,} \end{cases}$$

and notice that the variables X_i are iid. Their law is given by

- $\mathbb{P}(X_i = 1) = \mu(t)$;

- $\mathbb{P}(X_i = -1) = \mu\left(t^{-1}\right)$;
- $\mathbb{P}(X_i = 0) = \mu(b) + \mu(b^{-1}) = 1 - \left(\mu(t) + \mu\left(t^{-1}\right)\right)$.

For every $k \in \mathbb{N}$, one has:

$$\begin{aligned}\sum_{i=1}^{k} X_i &= \sum_{i=1}^{k} \mathbb{1}_{\{G_i = t\}} - \mathbb{1}_{\{G_i = t^{-1}\}} \\ &= \left|\{i \in [\![1,k]\!] \mid G_i = t\}\right| - \left|\{i \in [\![1,k]\!] \mid G_i = t^{-1}\}\right|.\end{aligned}$$

Thus, by Lemma 3.2 applied to the sequence $(X_i)_{i\in\mathbb{N}}$, we get the two following equalities:

(1)

$$\begin{aligned}\mathbb{P}\left(\left|\{i \in [\![1,k]\!] \mid G_i = t\}\right| > \left|\{i \in [\![1,k]\!] \mid G_i = t^{-1}\}\right|,\ \forall k \in \mathbb{N}\right) &= \mathbb{P}\left(\sum_{i=1}^{k} X_i > 0, \forall k \in \mathbb{N}\right) \\ &= \mu(t) - \mu\left(t^{-1}\right) \\ &> 0.\end{aligned}$$

(2)

$$\begin{aligned}&\mathbb{P}\left(\lim_{k\to\infty} \frac{\left|\{i \in [\![1,k]\!] \mid G_i = t\}\right| - \left|\{i \in [\![1,k]\!] \mid G_i = t^{-1}\}\right|}{k} = \mu(t) - \mu\left(t^{-1}\right)\right) \\ &= \mathbb{P}\left(\lim_{k\to\infty} \frac{\sum_{i=1}^{k} X_i}{k} = \mu(t) - \mu\left(t^{-1}\right)\right) \\ &= 1.\end{aligned}$$

For now, let us argue deterministically. Let $(g_k)_{k\in\mathbb{N}} \in \{b, b^{-1}, t, t^{-1}\}$ be a sequence of elements satisfying these conditions, *i.e.* with the notations of Proposition 3.1:

(1) $\mathfrak{h}_k^+ > \mathfrak{h}_k^-$ for every $k \in \mathbb{N}^*$;
(2) $\lim_{k\to\infty} \frac{\mathfrak{h}_k^+ - \mathfrak{h}_k^-}{k} = \mu(t) - \mu\left(t^{-1}\right)$

and let $s_k = g_1 \dots g_k$. For any subgroup $\Lambda \leq \Gamma$ such that $\Lambda \cap \langle b \rangle = \langle b^N \rangle$, one has $s_k^{-1}\Lambda s_k \cap \langle b \rangle = \langle b^{N_k} \rangle$ where

$$\begin{aligned}|N_k|_p &= (\mathfrak{h}_k^+ - \mathfrak{h}_k^-)(|m|_p - |n|_p) + |N|_p \quad \text{by the first condition and Proposition 3.1} \\ &> k\frac{\mu(t) - \mu\left(t^{-1}\right)}{2} \quad \text{for } k \geq k_0 \text{ large enough by the second condition}\end{aligned}$$

(where k_0 only depends on $(g_k)_{k\in\mathbb{N}}$). Thus, $|N_k|_p$ tends to ∞ as k goes to ∞. In particular,

$$\lim_{k\to\infty} s_k^{-1}\Lambda s_k \cap \langle b \rangle = \{1\}.$$

Hence, denoting by U_M the open subset $U_M = \left\{\Lambda \leq \Gamma \mid \Lambda \cap \langle b\rangle = \left\langle b^M\right\rangle\right\}$ of $\mathbf{Ph}_{m,n}^{-1}(P)$ (for any $M \in \mathrm{Ph}_{m,n}^{-1}(P)$), one has

$$s_k^{-1} U_N s_k \cap U_M = \emptyset \text{ as soon as } k > \max\left(k_0, \frac{2|M|_p}{\mu(t) - \mu\left(t^{-1}\right)}\right).$$

Thus, for any $M \in \mathrm{Ph}_{m,n}^{-1}(P)$:

$$\mathbb{P}\left(\exists k_0 : S_k^{-1} U_N S_k \cap U_M = \emptyset,\ \forall k \geq k_0\right) \geq \mu(t) - \mu\left(t^{-1}\right),$$

which is a strong negation of being topologically μ-mixing for the action by conjugation on $\mathbf{Ph}_{m,n}^{-1}(P)$. □

3.2. **Proof of Theorem 1.2.** In this section, we consider a probability measure $\mu : \Gamma \to [0,1]$ whose support is bounded, symmetric, and generates Γ. Let $(S_k)_{k\in\mathbb{N}} = (G_1 \dots G_k)_{k\in\mathbb{N}}$ be the random walk with step distribution μ.

We first prove that for any transitive action α on a pointed countable set (X, x) whose $\mathcal{G}$-graph $\mathcal{G}_\alpha$ is infinite, denoting by $p : \mathrm{Sch}(\alpha) \to \mathcal{G}_\alpha$ the projection that shrinks the $\langle b\rangle$-orbits (see Subsection 2.2.3), the image $(p(x \cdot S_k))_{k\in\mathbb{N}}$ almost surely escapes from every finite subgraph of $\mathcal{G}_\alpha$.

Lemma 3.4. *Let α be a transitive action on a pointed countable set (X, x), and let K be a finite subgraph of the (m, n)-graph $\mathcal{G}_\alpha$ of α such that the subgraph K^c induced by the set of vertices of $\mathcal{G}_\alpha \smallsetminus K$ is non empty. Let us denote by $p : \mathrm{Sch}(\alpha) \to \mathcal{G}_\alpha$ the canonical surjection. Then*

$$\mathbb{P}\left(p\left(x \cdot S_k\right) \in \mathcal{V}\left(K\right),\ \forall k \in \mathbb{N}\right) = 0.$$

Proof. We first build a finite set $\mathcal{F} \subseteq \Gamma$ satisfying the following property: for every $W \in \mathcal{V}(K)$, for every $y \in p^{-1}(W)$, there exists an element $f_y \in \mathcal{F}$ such that $p(y \cdot f_y) \notin \mathcal{V}(K)$. Let us consider a reduced edge path $E_1^W, \dots, E_{r_W}^W$ with source W and target outside K. Let $y \in p^{-1}(W)$ and γ_W be a reduced word in b, b^{-1}, t, t^{-1} such that the edge path $E_1^W, \dots, E_{r_W}^W$ derives from (γ_W, y). In particular, one has $W' := p(y \cdot \gamma_W) \notin \mathcal{V}(K)$. Let us define the integer

$$N_W = |mn|^{r_W}.$$

As the height of γ_W is equal to r_W, the following equality holds:

$$\gamma_W b^{N_W} = b^{N_W} \gamma_W.$$

In particular, for every integer N which is divisible by N_W, one has

$$\gamma_W b^N = b^N \gamma_W.$$

Let $z \in y \cdot \langle b\rangle = p^{-1}(W)$ and let $k_0 \in \mathbb{Z}$ such that $z \cdot b^{k_0} = y$. Up to replacing k_0 by the remainder of its division by N_W, we get that $z \cdot b^{k_0} = y \cdot b^{N_z}$ for some integer N_z divisible by

N_W and some integer k_0 satisfying $|k_0| \leq N_W$. Thus,

$$\begin{aligned} z \cdot b^{k_0}\gamma_W &= y \cdot b^{N_z}\gamma_W \\ &= y \cdot \gamma_W b^{N_z} \quad \text{because } N_z \text{ is divisible by } N_W \end{aligned}$$

which implies that

$$\begin{aligned} p(z \cdot b^{k_0}\gamma_W) &= p(y \cdot \gamma_W) \\ &= W' \notin \mathcal{V}(K). \end{aligned}$$

Hence the (finite) set

$$\mathcal{F} = \bigcup_{W \in \mathcal{V}(K)} \left\{ b^{k_0}\gamma_W \mid |k_0| \leq N_W \right\}$$

is suitable.

For every $f \in \mathcal{F}$, using the fact that $\mathrm{Supp}(\mu)$ is a symmetric and generating set of Γ, let us write $f = \mathfrak{g}_1^f \dots \mathfrak{g}_{N_f}^f$ for some $N_f \in \mathbb{N}$ and $\mathfrak{g}_i^f \in \mathrm{Supp}(\mu)$ for every $i \in [\![1, N_f]\!]$. Let us denote by

$$\theta = \min_{f \in \mathcal{F}} \prod_{i=1}^{N_f} \mu\left(\mathfrak{g}_i^f\right) \in]0,1[$$

and by

$$L = \max_{f \in \mathcal{F}} N_f.$$

Notice that for every $k \in \mathbb{N}$, the following holds:

$$\begin{aligned} &\mathbb{P}\left(p\left(x \cdot S_{k+i}\right) \in \mathcal{V}(K),\ \forall i \in [\![0, L-1]\!]\right) \\ &\leq \sum_{z \in p^{-1}(\mathcal{V}(K))} \mathbb{P}(p(x \cdot S_{k+i}) \in \mathcal{V}\left(K\right),\ \forall i \in [\![0, L-1]\!] \ \text{ and } \ x \cdot S_k = z) \\ &\leq \sum_{z \in p^{-1}(\mathcal{V}(K))} \mathbb{P}(\{G_{k+1} \dots G_{k+N_{f_z}} \neq f_z\} \cap \{x \cdot S_k = z\}) \\ &= \sum_{z \in p^{-1}(\mathcal{V}(K))} \mathbb{P}(\{G_{k+1} \dots G_{k+N_{f_z}} \neq f_z\})\mathbb{P}(\{x \cdot S_k = z\}) \ \text{ by independance of the } G_i\text{'s, } i \in \mathbb{N}^* \\ &\leq \sum_{z \in p^{-1}(\mathcal{V}(K))} (1-\theta)\mathbb{P}(\{x \cdot S_k = z\}) \\ &= (1-\theta)\mathbb{P}(p(x \cdot S_k) \in K) \\ &\leq 1 - \theta. \end{aligned}$$

Thus, a straightforward induction on $N \in \mathbb{N}$ (which relies on the independence of the G_i's) shows that:

$$\mathbb{P}(p(x \cdot S_k) \in \mathcal{V}\left(K\right),\ \forall k \in [\![0, NL-1]\!]) \leq (1-\theta)^N,$$

which tends to 0 as N tends to ∞. $\square$

To prove Theorem 1.2, we will make use of the following result, which is a consequence of the main theorem of [CS89]:

Theorem 3.5. *Let $(m,n) \in \mathbb{Z}^2$ such that $\min(|m|,|n|) > 1$. For every $v \in \mathcal{V}(\mathcal{T}_{m,n})$, the sequence $S_k \cdot v$ converges almost surely to a random end $\xi \in \partial\mathcal{T}_{m,n}$.*

Remark 3.6. As $d(S_k \cdot v, S_k \cdot w) = d(v,w)$, the sequence $(d(S_k \cdot v, S_k \cdot w))_{k\in\mathbb{N}}$ is bounded for every $v, w \in \mathcal{T}_{m,n}$, thus the limit does not depend on v.

Theorem 3.5 relies on the fact that $(S_k)_{k\in\mathbb{N}}$ is a regular random walk on the automorphism group $Aut(\mathcal{T}_{m,n})$ of the infinite locally finite tree $\mathcal{T}_{m,n}$ and that $\mathrm{Supp}(\mu)$ is not contained in any amenable closed subgroup of $Aut(\mathcal{T}_{m,n})$. For more details, see [CS15, Lemma 4.8]. Thus the hypotheses of [CS89, Theorem] are satisfied.

From this theorem and Lemma 3.4, we deduce the following result:

Proposition 3.7. *Let α be a transitive and non saturated preaction on a pointed countable set (X,x) whose (m,n)-graph K is finite, and let β be the maximal forest saturation action of α (defined on a pointed countable set (X',x) that contains X). Let $\Lambda = \mathrm{Stab}_\alpha(x)$. Let $\mathcal{G}$ be the (m,n)-graph of β. Let us denote by $\pi : \mathcal{T}_{m,n} \to \mathcal{G}$ the projection and let $\mathcal{T}^{\Lambda}_{m,n} = \pi^{-1}(K)$ be the minimal Λ-invariant subtree of $\mathcal{T}_{m,n}$. Then for every vertex $v \in \mathcal{V}(\mathcal{T}_{m,n})$, the random walk $S_k \cdot v$ converges almost surely to a random end $\xi \in \partial\mathcal{T}_{m,n} \smallsetminus \partial\mathcal{T}^{\Lambda}_{m,n}$.*

Proof. By Remark 2.6, $\mathcal{T}^{\Lambda}_{m,n}$ is a proper Λ-invariant subtree of $\mathcal{T}_{m,n}$. The fact that the random walk $S_k \cdot v$ converges almost surely to a random end $\xi \in \partial\mathcal{T}_{m,n}$, and that the limit does not depend on v, results from Theorem 3.5. So one can assume that $v = \langle b\rangle$. It remains to show that $\xi \notin \partial\mathcal{T}^{\Lambda}_{m,n}$.

Let us denote by $p : X' \to \mathcal{G}$ the projection and let $V := p(x)$. As $\mathcal{G}$ is infinite, applying Lemma 3.4 to the compact subgraph $B_{\mathcal{G}}(V,N)$ (the ball of center V and radius N) for every $N \in \mathbb{N}$ tells us that the distance in $\mathcal{G}$ between $p(x \cdot S_k)$ and V is unbounded asymptotically almost surely. Hence, from

$$(\Lambda G_1 \dots G_k)\langle b\rangle = p(x \cdot S_k)$$

and

$$\Lambda(G_1 \dots G_k\langle b\rangle) = \pi(S_k \cdot v),$$

we get that

$$p(x \cdot S_k) = \pi(S_k \cdot v),$$

so the distance between V and $\pi(S_k \cdot v)$ is unbounded asymptotically almost surely. Thus, the limit ξ of $S_k \cdot v$ is in $\partial\mathcal{T}_{m,n} \smallsetminus \partial\mathcal{T}^{\Lambda}_{m,n}$ asymptotically almost surely. $\square$

In particular, we get the following corollary in the setting of Theorem 1.2:

Corollary 3.8. *Let α be a transitive and non saturated preaction on a pointed countable set (X, x) whose (m, n)-graph K is finite and let β be its maximal forest saturation action (defined on a pointed countable set (X', x) that contains X). Let $\Lambda = \mathrm{Stab}_\alpha(x)$ and let $\mathcal{G}$ be the (m, n)-graph of β. Let us assume that*

- *either $|m| = |n|$; or*
- *α has infinite phenotype.*

Then, denoting by $p : X' \to \mathcal{G}$ the canonical projection, the sequence $p(x \cdot S_k)$ converges almost surely to a random end of $\mathcal{G}$.

Proof. Let $\mathcal{T}_{m,n}^{\Lambda} = \pi^{-1}(K)$. By Proposition 3.7, the sequence $(S_k \cdot v)_{k\in\mathbb{N}}$ converges almost surely to a random end $\xi \in \partial\mathcal{T}_{m,n} \smallsetminus \partial\mathcal{T}_{m,n}^{\Lambda}$. Let $e \in \mathcal{E}(\mathcal{T}_{m,n}) \smallsetminus \mathcal{E}\left(\mathcal{T}_{m,n}^{\Lambda}\right)$ such that ξ belongs to the half-tree $\widehat{\mathcal{T}_{m,n}}$ of e. For k large enough, the sequence $S_k \cdot v$ remains in $\widehat{\mathcal{T}_{m,n}}$ with high probability. By Proposition 2.7, the projection induces a homeomorphism $\pi : \widehat{\mathcal{T}_{m,n}} \to \pi\left(\widehat{\mathcal{T}_{m,n}}\right)$, thus the sequence $\pi(S_k \cdot v) = p(x \cdot S_k)$ converges almost surely to the random end $\pi(\xi)$ in $\pi\left(\widehat{\mathcal{T}_{m,n}}\right)$. □

The proof of Theorem 1.2 relies on the following key deterministic result:

Lemma 3.9. *Let α_1 and α_2 be transitive non saturated preactions on pointed countable sets (X_i, x_i) $(i \in \{1, 2\})$ whose (m, n)-graphs K_1 and K_2 share the same phenotype P. Assume that this phenotype is infinite if $|m| \neq |n|$. For $i \in \{1, 2\}$, let β_i be the maximal forest saturation action of α_i (defined on a pointed coutable set (Y_i, x_i) that contains X_i) and let $\mathcal{G}_i$ be the (m, n)-graph of β_i. Let us denote by $p_i : Y_i \to \mathcal{G}_i$ the projection.*

Let us consider reduced words s_1, s_2, s_3 such that:

1. *for every subword w of s_2s_3, one has $p_1(x_1 \cdot s_1 w) \notin K_1$;*
2. *for every subword w of $s_2^{-1}s_1^{-1}$, one has $p_2(x_2 \cdot s_3^{-1} w) \notin K_2$;*
3. *one has*

$$d_{\mathcal{G}_1}(p_1(x_1 \cdot s_1), p_1(x_1 \cdot s_1 s_2)) \geq d_{\mathcal{G}_1}(K_1, p_1(x_1 \cdot s_1)) + d_{\mathcal{G}_2}\left(K_2, p_2\left(x_2 \cdot s_3^{-1}\right)\right) + 2.$$

Then, there exists an action α, whose (m, n)-graph is infinite, that extends both α_1 and α_2, and such that $x_1 \cdot s_1 s_2 s_3 = x_2$.

Proof. Let us define the subset

$$X_1' = X_1 \cup \{p_1^{-1}(p_1(x_1 \cdot w)) \mid w \text{ subword of } s_1\}$$

of Y_1 and the subset

$$X_2' = X_2 \cup \{p_2^{-1}(p_2(x_2 \cdot w)) \mid w \text{ subword of } s_3^{-1}\}$$

of Y_2. For $i \in \{1, 2\}$, let us denote by β_i' the restriction of β_i defined on X_i'.

As $P = \infty$ or $|m| = |n|$, the following arguments will take place in a half-tree of K_i^c (for $i \in \{1, 2\}$), which is homeomorphic to a half-tree of $\mathcal{T}_{m,n}$ by Proposition 2.7; in particular, an edge path of K_i^c that derives from a vertex of K_i^c and a reduced word γ is reduced, and its length is the height $\mathfrak{h}(\gamma)$ of γ.

Especially, the edge path deriving from $(s_2, x_1 \cdot s_1)$ is reduced in K_1^c. Thus, one can write the normal form of s_2 as $s_2 = uv$, where all the subwords of u are defined on $x_1 \cdot s_1$ and no nonempty subword of v is defined on $x_1 \cdot s_1 u$ (for the preaction β_1'). One has:

$$\begin{aligned}\mathfrak{h}(u) &= d_{\mathcal{G}_1}(p_1(x_1 \cdot s_1), p_1(x_1 \cdot s_1 u)) \\ &\leq d_{\mathcal{G}_1}(p_1(x_1 \cdot s_1), K_1).\end{aligned}$$

Likewise, one can write the reduced normal form of s_2 as $s_2 = u'v'$, where all subwords of v'^{-1} are defined on $x_2 \cdot s_3^{-1}$ and no nonempty subword of u'^{-1} is defined on $x_2 \cdot s_3^{-1} v'^{-1}$ (for the preaction β_2'). One has

$$\begin{aligned}\mathfrak{h}(v') &= d_{\mathcal{G}_2}\left(p_2\left(x_2 \cdot s_3^{-1}\right), p_2\left(x_2 \cdot s_3^{-1} v'^{-1}\right)\right) \\ &\leq d_{\mathcal{G}_2}\left(p_2\left(x_2 \cdot s_3^{-1}\right), K_2\right).\end{aligned}$$

Thus, the third assumption implies that

$$\begin{aligned}\mathfrak{h}(s_2) &= d_{\mathcal{G}_1}(p_1(x_1 \cdot s_1), p_1(x_1 \cdot s_1 s_2)) \\ &\geq \mathfrak{h}(u) + \mathfrak{h}(v') + 2,\end{aligned}$$

so the initial subword u of s_2 is in fact an initial subword of u', and one can write the normal form of s_2 as $s_2 = uu''v'$, where $\mathfrak{h}(u'') \geq 2$.

Let us write $u'' = t^{\varepsilon} \mathfrak{m} t^{\eta}$ for some reduced word $\mathfrak{m}$, where

- $\varepsilon, \eta \in \{1, -1\}$;
- t^{ε} is not defined on $x_1 \cdot s_1 u$ (for the preaction β_1');
- $t^{-\eta}$ is not defined on $x_2 \cdot s_3^{-1} v'^{-1}$ (for the preaction β_2').

Let us denote by $b^{n_1} t^{\varepsilon_1} \dots b^{n_r} t^{\varepsilon_r} b^{n_{r+1}}$ the reduced form of $\mathfrak{m}$. By induction on r, we build a preaction γ defined on a countable set S

- whose (m, n)-graph is an edge path $E_1, \dots, E_r$, such that the orientation of the edge E_i is the sign of ε_i;
- all of whose $\langle b \rangle$-orbits share the same cardinal C, which is the common label of the vertices of K_i^c for $i \in \{1, 2\}$ by Remark 2.12 (and which is infinite if P is);
- such that there exist $y_1, y_2 \in S$ such that
  - $t^{-\varepsilon}$ is not defined on y_1;
  - t^{η} is not defined on y_2;
  - $y_2 = y_1 \cdot \mathfrak{m}$.

Finally, we merge the preactions β_1', γ and β_2' into a single preaction defined on $X_1' \sqcup S \sqcup X_2'$ by defining

- $x_1 \cdot s_1 u \cdot t^{\varepsilon} = y_1$;
- $y_2 \cdot t^{\eta} = x_2 \cdot s_3^{-1} v'^{-1}$.

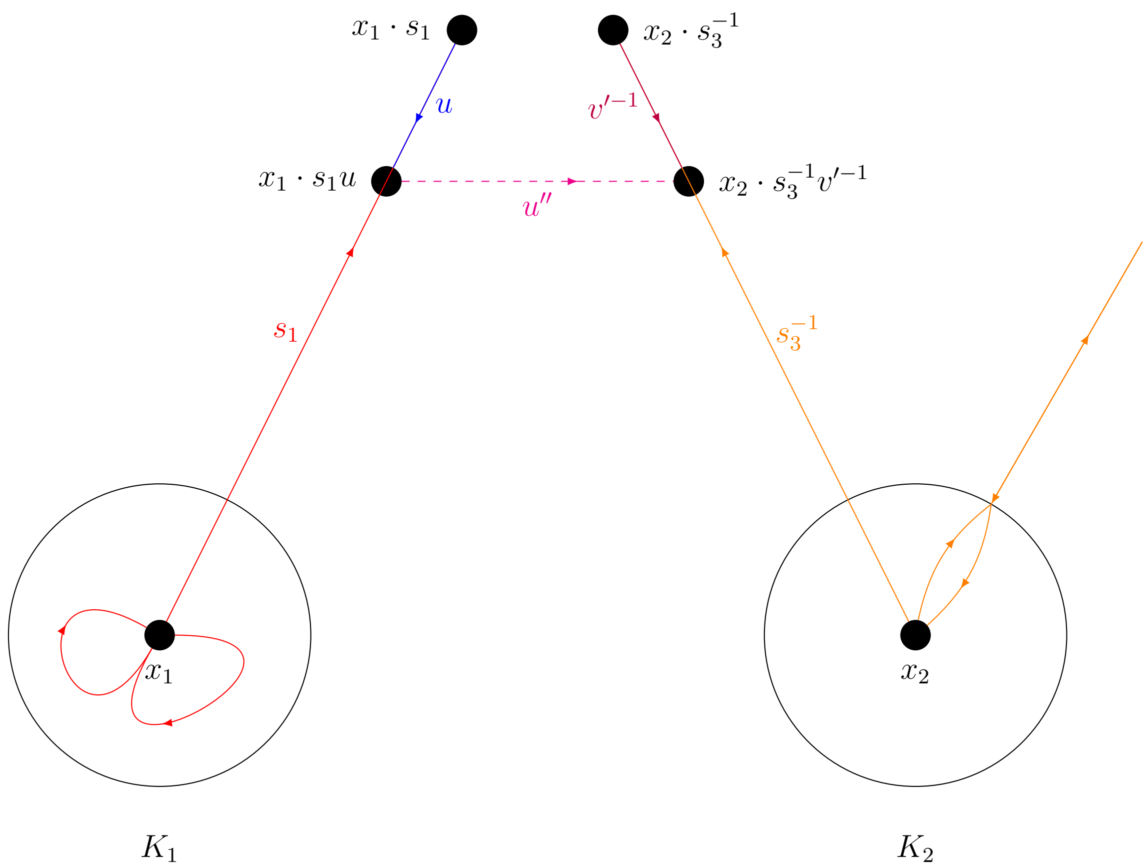

FIGURE 1. An illustration of the proof of Lemma 3.9

An illustration of this construction on the level of (m, n)-graphs is provided in Figure 1. For this new preaction δ we get

$$\begin{aligned} x_2 \cdot s_3^{-1} v'^{-1} &= y_2 \cdot t^\eta \\ &= y_1 \cdot \mathfrak{m} \cdot t^\eta \\ &= x_1 \cdot s_1 u t^\varepsilon \mathfrak{m} \cdot t^\eta \\ &= x_1 \cdot s_1 u u'' \end{aligned}$$

which implies that

$$\begin{aligned} x_2 &= x_1 \cdot s_1 u u'' v' s_3 \\ &= x_1 \cdot s_1 s_2 s_3 \end{aligned}$$

and δ extends both α_1' and α_2'. The (m, n)-graph of δ consists in

- the (m, n)-graph K_1' of β_1';

- the (m,n)-graph K_2' of β_2';
- the edge path $E_1, \dots, E_r$ (with $\mathbf{s}(E_1) \in K_1'$ and $\mathbf{t}(E_N) \in K_2'$), all of whose vertices are labeled C.

Thus, as $|m|, |n|, r \geq 2$, the vertex $\mathbf{t}(E_1)$ is not saturated. Hence, the maximal forest saturation action α of δ given by Lemma 2.4 has an infinite (m,n)-graph, thus satisfies the required conditions. $\square$

We are now ready to prove Theorem 1.2:

Proof of Theorem 1.2. Let Λ_1 and Λ_2 be two subgroups of Γ whose (m,n)-graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are infinite. Let us assume that

- either $\mathbf{Ph}_{m,n}(\Lambda_i) = \infty$ for $i \in \{1,2\}$; or
- $|m| = |n|$ and $\mathbf{Ph}_{m,n}(\Lambda_1) = \mathbf{Ph}_{m,n}(\Lambda_2)$,

and let us denote by P the common phenotype of Λ_1 and Λ_2.

For $i \in \{1,2\}$, let us denote by α_i the associated pointed transitive right action on a pointed countable set (X_i, x_i). Let $q_i : X_i \to \mathcal{G}_i$ be the canonical surjection and let $V_i := q_i(x_i)$. Let us fix $R > 0$ and let K_i be the R-ball of $\mathcal{G}_i$ around V_i. We denote by

$$U_i^{(R)} = \{\Lambda \in \mathcal{K}(\Gamma) \mid \text{the (labelled) } R\text{-ball of the pointed transitive action associated to } \Lambda \text{ is } K_i\}.$$

We recall that the sets $\left(U_i^{(R)}\right)_{R>0}$ form a basis of neighbourhoods of Λ_i for the topology on the set of pointed transitive actions defined in Subsection 2.2.3 induced on $\mathcal{K}(\Gamma)$. Moreover, $U_i^{(R)}$ is included in $\mathbf{Ph}_{\boldsymbol{m},\boldsymbol{n}}^{-1}(P)$.

Let α_i' be the subpreaction of α_i defined on a subset $X_i' \subseteq X_i$ containing x_i and whose (m,n)-graph is K_i. For $i \in \{1,2\}$, let β_i be the maximal forest saturation action of α_i given by Lemma 2.4 (defined on a countable set Y_i that contains X_i') and let $\mathcal{G}_i'$ be its (m,n)-graph. Let us denote by $p_i : Y_i \to \mathcal{G}_i'$ the projection.

Let us fix $\varepsilon > 0$. Using the fact that $\mathrm{Supp}(\mu)$ is bounded, let us denote by $M := \max_{\gamma \in \mathrm{Supp}(\mu)} \mathfrak{h}(\gamma)$. For $i \in \{1,2\}$, let $K_{i,M}$ be the M-neighborhood of K_i in $\mathcal{G}_i'$. By Corollary 3.8 applied to β_1 and the random walk $S_k = G_1 \dots G_k$ on the one hand, and to β_2 and the reversed random walk $S_k^{-1} = G_k^{-1} \dots G_1^{-1}$ on the other hand (legit, because μ is symmetric), we get that

$$\mathbb{P}\left(\exists k_0 \in \mathbb{N} : p_1(x_1 \cdot S_k) \notin \mathcal{V}(K_{1,M}),\ \forall k \geq k_0\right) = 1 \tag{8}$$

and

$$\mathbb{P}\left(\exists k_0 \in \mathbb{N} : p_2(x_2 \cdot S_k^{-1}) \notin \mathcal{V}(K_{2,M}),\ \forall k \geq k_0\right) = 1. \tag{9}$$

For every $k_0 \in \mathbb{N}$, let us define the events

$$A_{k_0} = \bigcap_{k > 2k_0} \{p_1(x_1 \cdot G_1 \dots G_{k_0} U) \notin \mathcal{V}(K_1) \ \text{ for every subword } U \text{ of } G_{k_0+1} \dots G_k\}$$

and

$$B_{k_0} = \bigcap_{k>2k_0} \left\{ p_2(x_2 \cdot G_k^{-1} \dots G_{k-k_0+1}^{-1} U) \notin \mathcal{V}(K_2) \text{ for every subword } U \text{ of } G_{k-k_0}^{-1} \dots G_1^{-1} \right\}.$$

We draw the attention of the reader to the fact that the $G_i's$ may not be one of the standard generators b, b^{-1}, t, t^{-1}, and that the term *subword* has to be understood in the sense of Section 2.2.2, *i.e.* with respect to the standard generators.

By (8) and (9), the choice of M implies that

$$\mathbb{P}\left(\bigcup_{k_0 \in \mathbb{N}} A_{k_0}\right) = 1$$

and

$$\mathbb{P}\left(\bigcup_{k_0 \in \mathbb{N}} B_{k_0}\right) = 1.$$

Let us fix $k_0 \in \mathbb{N}$ such that

$$\mathbb{P}(A_{k_0}) \geq 1 - \varepsilon \tag{10}$$

and

$$\mathbb{P}(B_{k_0}) \geq 1 - \varepsilon. \tag{11}$$

Let us define $M_{k_0} := \max_{\left(h_1, \dots, h_{k_0}\right) \in \mathrm{Supp}(\mu)^{k_0}} \mathfrak{h}(h_1 \dots h_{k_0})$ and, for $i \in \{1, 2\}$, let us define

$$C_i = \max_{\left(h_1, \dots h_{k_0}\right) \in \mathrm{Supp}(\mu)^{k_0}} d_{\mathcal{G}_i'}\left(p_i\left(x_i \cdot h_1 \dots h_{k_0}\right), K_i\right).$$

For every $k_0' \in \mathbb{N}$, let us define the event

$$D_{k_0'} = \left\{ d_{\mathcal{G}_1'}\left(p_1(x_1), p_1\left(x_1 \cdot G_1 \dots G_k\right)\right) \geq C_1 + C_2 + 2M_{k_0} + 2, \ \forall k > k_0' \right\}.$$

By Corollary 3.8, there exists $k_0' > 2k_0$ such that

$$\mathbb{P}\left(D_{k_0'}\right) \geq 1 - \varepsilon. \tag{12}$$

Thus, by Equations (10), (11) and (12), we get

$$\mathbb{P}\left(A_{k_0} \cap B_{k_0} \cap D_{k_0'}\right) \geq 1 - 3\varepsilon.$$

Let $(g_i)_{i \in \mathbb{N}} \in \mathrm{Supp}(\mu)^{\mathbb{N}}$ satisfying these three conditions, *i.e.*

- $p_1\left(x_1 \cdot g_1 \dots g_{k_0} u\right) \notin K_1$ for every subword u of $g_{k_0+1} \dots g_k$ and for every $k > 2k_0$;
- $p_2\left(x_2 \cdot g_k^{-1} \dots g_{k-k_0+1}^{-1} v\right) \notin K_2$ for every subword v of $g_{k-k_0}^{-1} \dots g_1^{-1}$ and for every $k > 2k_0$;
- $d_{\mathcal{G}_1'}\left(p_1(x_1), p_1\left(x_1 \cdot g_1 \dots g_k\right)\right) \geq C_1 + C_2 + 2M_{k_0} + 2$ for every $k > k_0'$.

Then, for every $k > k_0' > 2k_0$:

$$\begin{aligned} d_{\mathcal{G}_1'}\left(p_1(x_1 \cdot g_1 \dots g_{k_0}), p_1(x_1 \cdot g_1 \dots g_{k_0} \dots g_{k-k_0})\right) &\geq d_{\mathcal{G}_1'}\left(p_1(x_1), p_1(x_1 \cdot g_1 \dots g_k)\right) \\ &\quad - d_{\mathcal{G}_1'}\left(p_1(x_1), p_1(x_1 \cdot g_1 \dots g_{k_0})\right) \\ &\quad - d_{\mathcal{G}_1'}\left(p_1(x_1 \cdot g_1 \dots g_{k-k_0}), p_1(x_1 \cdot g_1 \dots g_k)\right) \\ &\quad \text{(by the triangle inequality)} \\ &\geq (C_1 + C_2 + 2M_{k_0} + 2) - M_{k_0} - M_{k_0} \\ &= C_1 + C_2 + 2 \\ &\geq d_{\mathcal{G}_1'}\left(K_1, p_1(x_1 \cdot g_1 \dots g_{k_0})\right) \\ &\quad + d_{\mathcal{G}_2'}\left(K_2, p_2\left(x_2 \cdot g_k^{-1} \dots g_{k-k_0+1}^{-1}\right)\right) + 2, \end{aligned}$$

which implies that the preactions α_1', α_2' and the reduced forms s_1, s_2, s_3 of the three elements $g_1 \dots g_{k_0}$, $g_{k_0+1} \dots g_{k-k_0}$ and $g_{k-k_0+1} \dots g_k$ of Γ satisfy the assumptions of Lemma 3.9. Thus, by Lemma 3.9, there exists a saturated preaction α defined on a pointed countable set (X, x_1) that contains X_1' and X_2' as disjoint subsets and such that

- α extends both α_1' and α_2';
- $x_1 \cdot g_1 \dots g_k = x_2$.

We proved that for every $R > 0$ and every $\varepsilon > 0$, there exists $k_0' \in \mathbb{N}$ such that, for every $k > k_0'$ one has

$$\mathbb{P}\left(\exists \Lambda \in \mathcal{K}(\Gamma) \cap \mathbf{Ph}_{m,n}^{-1}(P) : \Lambda \in U_1^{(R)} \cap S_k^{-1} U_2^{(R)} S_k\right) \geq 1 - 3\varepsilon.$$

Thus, as the sets $\left(U_i^{(R)}\right)_{R>0}$ form a basis of neighbourhoods of Λ_i for the topology on the set of pointed transitive actions, which is finer than the Chabauty topology (*cf.* Remark 2.8), the conjugation action is topologically transitive on $\mathbf{Ph}_{\boldsymbol{m},\boldsymbol{n}}^{-1}(P)$. $\square$

References

[AG23] Pénélope Azuelos and Damien Gaboriau. Perfect kernel and dynamics: from Bass-Serre theory to hyperbolic groups. *Mathematische Annalen*, 391(3):4733–4789, 2024.

[Bon24] Sasha Bontemps. Perfect kernel of generalized Baumslag-Solitar groups, 2024. arXiv: 2411.03221.

[BS62] Gilbert Baumslag and Donald Solitar. Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.*, 68(6):199-201, 1962.

[CGLMS22] Alessandro Carderi, Damien Gaboriau, François Le Maître, and Yves Stalder. On the space of subgroups of Baumslag-Solitar groups I: perfect kernel and phenotype, 2022. arXiv:2210.14990.

[CS89] Donald I. Cartwright and P. M. Soardi. Convergence to Ends for Random Walks on the Automorphism Group of a Tree. *American Mathematical Society* 107(3), 817-823, 1989.

[CS15] Johannes Cuno and Ecaterina Sava-Huss. Random walks on Baumslag-Solitar groups. *Israel Journal of Mathematics*, 228, 627-663, 2018.

[FM198] Benson Farb and Lee Mosher. A rigidity theorem for the solvable Baumslag-Solitar groups. *Inventiones* 131(2), 419-451, 1998.

[GLMS24] Damien Gaboriau, François Le Maître, and Yves Stalder. On the space of subgroups of Baumslag-Solitar groups II: High transitivity, 2024. arXiv:2410.23224.

[GPT+] Damien Gaboriau, Antoine Poulin, Anush Tserunyan, Robin Tucker-Drob, and Konrad Wróbel. Measure equivalence classification of Baumslag-Solitar groups. in preparation.

[HMO24] Michael Hull, Ashot Minasyan, and Denis Osin Subgroup mixing and random walks in groups acting on hyperbolic spaces. arXiv: 2407.16069

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Grad. Texts Math.* Berlin: Springer-Verlag, 1995.

[Mes72] Stephan Meskin. Non residually finite one-relator groups. *Transactions of the American Mathematical Society* 164, 105-114, 1972.

[Ser77] Jean-Pierre Serre. *Arbres, amalgames,* SL_2*. Rédigé avec la collaboration de Hyman Bass*, volume 46 of *Astérisque*. Société Mathématique de France (SMF), Paris, 1977.

[Why01] Kevin Whyte. The large scale geometry of the higher Baumslag-Solitar groups. *Geom. Funct. Anal* 11(6):1327-1343, 2001.

ENS-Lyon, Unité de Mathématiques Pures et Appliquées, 69007 Lyon, France
sasha.bontemps@ens-lyon.fr