Dynamics on the perfect kernel of non-amenable higher rank generalized Baumslag-Solitar groups

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Abstract

In this article, we study the space of subgroups of non-amenable generalized Baumslag-Solitar groups (GBS groups) of rank d, that is, groups acting cocompactly on an oriented tree with vertex and edge stabilizers isomorphic to $\mathbb{Z}^d$. Our results generalize the study of Baumslag-Solitar groups, and of GBS groups of rank 1. We give an explicit description of the perfect kernel of a non-amenable GBS group G of rank d and show the existence of a partition of the perfect kernel into a countably infinite set of pieces which are invariant under the action by conjugation of G, and such that each piece contains a dense orbit.

Keywords: higher rank generalized Baumslag-Solitar groups; space of subgroups; Schreier graphs; perfect kernel; topologically transitive actions; Bass-Serre theory.

MSC-classification: 37B; 20E06; 20E08.

1 Introduction

A generalized Baumslag-Solitar group (GBS group) of rank d is a group that acts cocompactly on an oriented tree such that the vertex and edge stabilizers are isomorphic to $\mathbb{Z}^d$. As a consequence of Bass-Serre theory, a generalized Baumslag-Solitar group is defined by a finite iteration of HNN extensions and amalgamated free products of $\mathbb{Z}^d$ over $\mathbb{Z}^d$.

GBS groups of rank $d > 1$ are a generalization of GBS groups of rank 1, which arise as a natural generalization of Baumslag-Solitar groups $\mathrm{BS}(m,n) = \langle b, t \mid tb^n t^{-1} = b^m \rangle$. Baumslag-Solitar groups were introduced in [BS62] to give the first examples of two generated finitely presented non-Hopfian groups. GBS groups have been widely studied in relation to various properties. In [Lev15], Levitt computed the minimal number of generators of a GBS group of rank 1. He studied their automorphism groups in [Lev07]. The classification up to quasi-isometry of GBS groups of rank 1 is known (see [Why01]) and the classification up to measure equivalence of Baumslag-Solitar groups has been announced by the authors of [GPT+]. In [LdGZS25], the authors determined which GBS groups (of arbitrary rank) are residually finite and which are LERF.

In this article, we will focus on the set of subgroups $\mathrm{Sub}(G)$ of a GBS group G from a topological point of view. This means that we are more interested in the topological structure of $\mathrm{Sub}(G)$, seen as a closed subset of the Cantor set $\{0,1\}^G$, than in the algebraic properties of the subgroups of G. Cantor-Bendixson theory (see [Kec95]) leads to a unique decomposition $\mathrm{Sub}(G) = \mathcal{K}(G) \sqcup C$ into a closed subspace without isolated points called the **perfect kernel** of G and a countable set C. As the action by conjugation of G induces a homeomorphism of $\mathrm{Sub}(G)$, the perfect kernel is G-invariant. We are interested in the computation of the perfect kernel and the dynamics induced by the action by conjugation of G on it.

The perfect kernel of any finitely generated abelian group is empty. In [CGP10], the authors classified the set of subgroups of all countable abelian groups up to homeomorphism. In [BGK12], the authors proved that the perfect kernel of the lamplighter group $(\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z}$ (where p is a prime number) is the set of subgroups of $\bigoplus_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}^n$.

If G is a finitely generated group, then finite index subgroups are isolated. Thus the perfect kernel of G is included in the set of infinite index subgroups of G. The authors of [CGLM23] observed that equality holds for the non-abelian finitely generated free group $\mathbb{F}_r$ on r generators, and that the action of $\mathbb{F}_r$ on its perfect kernel is topologically transitive. Recall that an action of a group G on a topological space X is **topologically transitive** if for every non-empty open subsets $U, V \subseteq X$, there exists $g \in G$ such that $gU \cap V \neq \varnothing$. In the case where X is Polish, this is equivalent to the existence of a dense orbit. The authors of [AG24] extended this result to a large class of groups acting on trees. They proved that the perfect kernel of a finitely generated group G with infinitely many ends is also equal to the set of infinite index subgroups, and that the action by conjugation is topologically transitive on the perfect kernel as soon as G does not contain any non-trivial finite normal subgroup. More generally, they proved that for any finitely generated group G that acts (minimally and irreducibly) on a tree $\mathcal{T}$ such that the action of G on $\mathcal{T}$ is acylindrical, then any subgroup H of G satisfying that the quotient graph $H\backslash\mathcal{T}$ is infinite belongs to the perfect kernel of G. In this case, they also proved that the action by conjugation of G on the closure of the set of subgroups H acting on $\mathcal{T}$ with infinitely many orbits of edges is topologically transitive. Recall that an action of a group G on a tree $\mathcal{T}$

is **acylindrical** if there exists $R > 0$ such that the stabilizer of any path of length larger than R is trivial.

GBS groups of rank d are typical examples of groups whose action on their Bass-Serre tree is *not* acylindrical, because the stabilizer of any finite subtree of the Bass-Serre tree is isomorphic to $\mathbb{Z}^d$. The authors of [CGMS25] (who studied Baumslag-Solitar groups) and of [Bon24] (who extended some results obtained by the aforementioned authors to GBS groups of rank 1) observed that this leads to very different dynamics for the action by conjugation of a non-amenable GBS group G of rank 1 on its perfect kernel. More precisely, they showed that $\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G)$ if and only if G is not **unimodular**. Recall that one characterization of unimodularity for a GBS group of rank 1 is the existence of an infinite cyclic normal subgroup (see [Lev07][Section 2] for instance). They also described a countably infinite G-invariant partition of $\mathcal{K}(G)$, such that G acts topologically transitively on each piece. One piece is closed and all the other ones are open (and also closed if and only if G is unimodular). To obtain this decomposition, the authors of [CGMS25] introduced the **phenotype**, which is a G-invariant function $\mathrm{Sub}(G) \to \mathbb{N}^* \cup \{\infty\}$, and which was generalized in [Bon24]. This function is computable and encodes the decomposition of the perfect kernel.

In this article, we will extend these results to non-amenable GBS groups of an arbitrary rank d, that is to say, those which are neither isomorphic to $\mathbb{Z}[\mathrm{A}, \mathrm{A}^{-1}](\mathbb{Z}^d) \rtimes \mathbb{Z}$ for some $\mathrm{A} \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{Q})$ (where $\mathbb{Z}[\mathrm{A}, \mathrm{A}^{-1}](\mathbb{Z}^d)$ is the subgroup of $\mathbb{Q}^d$ defined by $\{\mathrm{A}^k u, k \in \mathbb{Z}, u \in \mathbb{Z}^d\}$ and $\mathbb{Z}$ acts on $\mathbb{Z}[\mathrm{A}, \mathrm{A}^{-1}](\mathbb{Z}^d)$ by multiplication by A) nor to any amalgamated free product $\mathbb{Z}^d *_{\mathbb{Z}^d} \mathbb{Z}^d$ where both injections are defined by matrices of determinant ± 2 (*cf.* Proposition 3.18). More precisely, we prove the following result (*cf.* Theorem 5.2):

Theorem 1.1. *Let G be a non-amenable GBS group of rank d defined by a reduced graph of groups $\mathscr{H}$ and let $\mathcal{T}$ be the Bass-Serre tree of $\mathscr{H}$. Then*

$$\mathcal{K}(G) = \{H \leq G \mid H \backslash \mathcal{T} \text{ is infinite}\}.$$

We also give some sufficient conditions that depend on the modular homomorphism (see Section 3.2) for the perfect kernel to be equal to $\mathrm{Sub}_{[\infty]}(G)$.

We also obtain a generalization of the main results of [CGMS25] and [Bon24] (*cf.* Equation 6.4 and Theorem 6.5 if G is not a semidirect product, and Theorem 6.15 otherwize):

Theorem 1.2. *Let G be a non-amenable GBS group of rank d. There exists a countably infinite G-invariant partition of the perfect kernel of G into pieces that contain dense orbits.*

This implies in particular that the action is topologically transitive on each piece.

We also investigate the topology of the pieces that appear in these decompositions, which is slightly different depending on whether G is a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$ (*cf.* Theorem 6.15) or not (*cf.* Proposition 6.6). As some of these pieces need not be Polish, proving high transitivity does not suffice to get the existence of a dense orbit in each piece.

The paper is organized as follows. Given a graph of groups $\mathscr{H}$ of fundamental group G, we extend the notion of $\mathscr{H}$-preactions and of $\mathscr{H}$-graphs in Section 3.1. They were introduced in [FLMMS22] in the case where G is an amalgamated free product or an HNN-extension (*i.e.* if $\mathscr{H}$ consists of a single edge), and adapted in [CGMS25] in the case of Baumslag-Solitar groups and in [Bon24] in the case of GBS groups of rank 1. In Section 5, we prove Theorem 1.1. Finally, in Section 6, we show the existence of the decomposition of Theorem 1.2 and investigate the topology of the pieces. This gives rise to a natural generalization of the phenotype defined in [CGMS25] and in [Bon24]. However, we do not know if this decomposition is still computable in this wider context, and we do not know if our arguments can be used to prove *high* topological transitivity results as in [GMS24] for Baumslag-Solitar groups and in [Bon24] for GBS groups of rank 1.

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Contents

2 Preliminaries and notations

We denote by $\mathcal{P}$ the set of prime numbers in $\mathbb{N}$. For every integer N and $p \in \mathcal{P}$, we denote by $|N|_p$ the p-adic valuation of N, that is, the largest $n \in \mathbb{N}$ such that p^n divides N. By "countable" we mean finite or in bijection with $\mathbb{N}$. Given a group G, we denote by $\mathrm{Sub}(G)$ the set of subgroups of G and by $\mathrm{Sub}_{[\infty]}(G)$ the subset of $\mathrm{Sub}(G)$ that consists of infinite index subgroups. If $H \leq G$ is a subgroup, we denote by $[H]_G$ the G-conjugacy class of H. For any $d \in \mathbb{N}^*$, we denote by $\mathcal{L}(\mathbb{Z}^d)$ the set of lattices of $\mathbb{Z}^d$, *i.e.* the set of finite index subgroups of $\mathbb{Z}^d$. If $\mathbb{K}$ is a field and $\mathrm{M} \in M_d(\mathbb{K})$, one denotes by $\mathrm{Spec}_{\mathbb{K}}(\mathrm{M})$ the spectrum of M in $\mathbb{K}$, *i.e.* the set of elements $\lambda \in \mathbb{K}$ such that $\det(\mathrm{M} - \lambda I_d) = 0$.

2.1 Graphs

We refer to [Bon24], Section 2 for the notations and definitions around graphs and Schreier graphs. We add the following terminology: given a graph $\mathscr{H}$, an element of $\mathcal{E}(\mathscr{H}) \times \{\mathbf{s}, \mathbf{t}\}$ is called an **half-edge**. The **inferior** half-edge of $e \in \mathcal{E}(\mathscr{H})$ is $(e, \mathbf{s})$ and its **superior** half-edge is $(e, \mathbf{t})$. Given a graph $\mathscr{H}$ and a spanning tree $\mathscr{T}$ of $\mathscr{H}$, for any vertices $u, v \in \mathcal{V}(\mathscr{H})$, we denote by $[u, v]_{\mathscr{T}}$ the unique edge path in $\mathscr{T}$ that connects u to v.

2.2 Space of subgroups of a countable group

Let G be an infinite countable group. Endowed with the **Chabauty topology**, the set of subgroups $\mathrm{Sub}(G)$ of G is a closed subspace of the Cantor set $\{0,1\}^G$. An explicit basis of open sets is given by the following clopen sets:

$$\mathcal{V}(O, I) = \{H \in \mathrm{Sub}(G), H \cap O = \varnothing \ \text{ and } \ I \subseteq H\}$$

for any finite subsets $O, I \subseteq G$.

We will make use of the following lemma, that describes the topology of some subsets of $\mathrm{Sub}(G)$.

Lemma 2.1. *Let G be a countable group and let G_0 be any subgroup of G. Then, for any subgroup $H_0 \leq G_0$, the set*

$$\{H \leq G \mid H \cap G_0 = H_0\}$$

is closed in $\mathrm{Sub}(G)$. *If moreover the group H_0 is finitely generated and has finite index in G_0, then it is also open.*

Proof. One has

$$\{H \leq G \mid H \cap G_0 = H_0\} = \bigcap_{(h,g)\in H_0\times G_0\smallsetminus H_0} \{H \leq G \mid h \in H \quad \text{and} \quad g \notin H\}$$

which is closed as an intersection of basic clopen subsets of $\mathrm{Sub}(G)$.

If H_0 is finitely generated and has finite index in G_0, let $\{h_1, \ldots, h_n\}$ be a finite generating set of H_0 and let us write $G_0/H_0 = \{H_0, g_1H_0, \ldots, g_mH_0\}$. We then have

$$\{H \leq G \mid H \cap G_0 = H_0\} = \bigcap_{(i,j)\in[\![1,n]\!]\times[\![1,m]\!]} \{H \leq G \mid h_i \in H \quad \text{and} \quad g_j \notin H\}$$

which is open as a finite intersection of basic clopen sets. $\square$

Applying Cantor-Bendixson Theorem (see [Kec95, Section 6, Chapter 1] for instance) to the Polish space $\mathrm{Sub}(G)$ leads to a unique decomposition $\mathrm{Sub}(G) = K \bigsqcup C$ where C is countable and K is a closed subspace of $\mathrm{Sub}(G)$ without isolated points. The set K is called the **perfect kernel** of G and denoted by $\mathcal{K}(G)$. It is the largest closed subset of $\mathrm{Sub}(G)$ without isolated points, or equivalently, the set of subgroups all of whose neighborhoods are uncountable.

If G is finitely generated, then finite index subgroups are isolated. In particular, we get the following inclusion $\mathcal{K}(G) \subseteq \mathrm{Sub}_{[\infty]}(G)$. The converse inclusion is true in the case of finitely generated free groups (see [CGLM23][Proposition 2.1] and [AG24][Corollary 5.17]):

Proposition 2.2. *Let $\mathbb{F}_r$ be the free group on r generators ($2 \leq r \leq \infty$). Then*

- *if $r < \infty$, then $\mathcal{K}(\mathbb{F}_r) = \mathrm{Sub}_{[\infty]}(\mathbb{F}_r)$;*
- *$\mathcal{K}(\mathbb{F}_\infty) = \mathrm{Sub}(\mathbb{F}_\infty)$.*

Moreover, there exists a dense orbit for the action by conjugation of $\mathbb{F}_r$ on $\mathcal{K}(\mathbb{F}_r)$.

Remark 2.3. The key point of the proof of Proposition 2.2 is the identification of subgroups of $\mathbb{F}_r = \langle (a_i)_{i\in[\![1,r]\!]}\rangle$ with coverings of the bouquet B_r of r circles (which are exactly the Schreier graphs of subgroups of $\mathbb{F}_r$ with respect to the generating set $\{a_i, i \in [\![1,r]\!]\}$). More precisely, this relies on the two following facts. Let B be a (possibly infinite) graph. Then:

- for every covering $E \to B$ and every connected finite subgraph $K \subsetneq E$, there exists a covering $p : E' \to B$ whose degree is infinite and such that E' contains a subgraph K' which is isomorphic to K as a labeled graph (this allows to obtain the aforementioned explicit description of the perfect kernel);

- for every coverings $E_i \to B$ ($i \in \mathbb{N}$), given any connected finite subgraph $K_i \subsetneq E_i$ (for every $i \in \mathbb{N}$), there exists a covering $p : E \to B$ whose degree is infinite and such that E contains disjoint subgraphs $(K_i')_{i \in \mathbb{N}}$ such that K_i' and K_i are isomorphic as labeled graphs for every $i \in \mathbb{N}$ (this allows to build a dense orbit).

This point of view will be useful in the study of the action by conjugation of a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$ on its perfect kernel (*cf.* Section 6.2).

2.3 Graphs of groups

In this section, we recall the fundamentals of Bass-Serre theory. We refer to [Ser83] for more details. A **graph of groups** is an oriented graph $\mathscr{H}$ equipped with a collection of **vertex groups** $G_v, v \in \mathcal{V}(\mathscr{H})$, a collection of **edge groups** $G_e, e \in \mathcal{E}(\mathscr{H})$ such that $G_e = G_{\bar{e}}$ for every edge $e \in \mathcal{E}(\mathscr{H})$ and, for $\mathbf{u} \in \{\mathbf{s}, \mathbf{t}\}$, injective homomorphisms $\iota_{e,\mathbf{u}} : G_e \hookrightarrow G_{\mathbf{u}(e)}$ such that $\iota_{e,\mathbf{s}} = \iota_{\bar{e},\mathbf{t}}$ for every edge e.

The **fundamental group** of a graph of groups $\mathscr{H}$ is defined by the following presentation: let us fix a spanning tree $\mathscr{T}$ in $\mathscr{H}$. Denote by $\{t_e, e \in \mathcal{E}(\mathscr{H})\}$ a generating set of the free group $\mathbb{F}_{|\mathcal{E}(\mathscr{H})|}$ of rank $|\mathcal{E}(\mathscr{H})|$ and define

$$G = \left(*_{v \in \mathcal{V}(\mathscr{H})} G_v * \mathbb{F}_{|\mathcal{E}(\mathscr{H})|} \right) / \left\langle\!\left\langle \left(t_e^{-1} \iota_{e,\mathbf{s}(e)}(x) t_e \iota_{e,\mathbf{t}(e)}(x)^{-1} \right)_{(e,x) \in \mathcal{E}(\mathscr{H}) \times G_e}, (t_e t_{\bar{e}})_{e \in \mathcal{E}(\mathscr{H})}, (t_e)_{e \in \mathcal{E}(\mathscr{T})} \right\rangle\!\right\rangle \tag{2.4}$$

The isomorphism class of the group G defined as above does not depend on the choice of the spanning tree (*cf.* Proposition 20 in [Ser83], Section 5.1).

There exists a (unique up to unique isomorphism) oriented tree $\mathcal{T}$, called **Bass-Serre tree** of $\mathscr{H}$ on which G acts without inversion with quotient $\mathscr{H}$ and such that there exist sections $\mathcal{V}(\mathscr{H}) \to \mathcal{V}(\mathcal{T})$ and $\mathcal{E}(\mathscr{H}) \to \mathcal{E}(\mathcal{T})$ (which we denote by $v \to \tilde{v}$ and $e \to \tilde{e}$ respectively) of the projection $\pi : \mathcal{T} \to \mathscr{H}$ satisfying the following conditions:

$$\mathrm{Stab}(\tilde{v}) = G_v \ \forall v \in \mathcal{V}(\mathscr{H}). \tag{2.5}$$

$$\mathrm{Stab}(\tilde{e}) = G_e \ \forall e \in \mathcal{E}(\mathscr{H}). \tag{2.6}$$

More precisely (*cf.* [Ser83][Section 5.3]), the set of vertices of $\mathcal{T}$ is

$$\mathcal{V}(\mathcal{T}) = \bigsqcup_{v \in \mathcal{V}(\mathscr{H})} G/G_v,$$

and its set of edges is

$$\mathcal{E}(\mathcal{T}) = \bigsqcup_{e\in\mathcal{E}(\mathscr{H})} G/G_e.$$

Conversely, any group action $G \curvearrowright \mathcal{T}$ without inversion is obtained by this construction (*cf.* [Ser83, Section 5]).

3 $\mathscr{H}$-preactions and $\mathscr{H}$-graphs

In this section we generalize the interpretation of graphs of subgroups as "blown up and shrunk" Schreier graphs obtained for HNN-extensions or amalgamated free products in [FLMMS22], for Baumslag-Solitar groups in [CGMS25] and for rank 1 GBS groups in [Bon24].

This gives us a tool to approximate some subgroups of iterated HNN-extensions and amalgamated free products, that we will apply to GBS groups.

3.1 General setting

In this section, we fix a graph of groups $\mathscr{H}$ endowed with a spanning tree $\mathscr{T}$ and we denote by G the fundamental group of $\mathscr{H}$, defined by Presentation (2.4). To any subgroup of G, we will associate a "$\mathscr{H}$-graph", which is a labelled graph that satisfy some combinatorial conditions. It will reduce the problem of approximating a subgroup of G to the one of approximating its $\mathscr{H}$-graphs.

First, we introduce the notion of **$\mathscr{H}$-preaction** of G. Informally, this is a collection of partial bijections (each of these corresponding to an edge generator or an element of a vertex group in the presentation (2.4)) such that the partial bijections associated to generators of a vertex group G_v define a genuine G_v-action. Let us make this definition more precise:

Definition 3.1. A $\mathscr{H}$-preaction on a countable set X is a collection of (possibly non-transitive) right G_v-actions α_v defined on subsets D_v of X for every $v \in \mathcal{V}(\mathscr{H})$ (*i.e.* morphisms $\alpha_v : G_v \to \mathrm{Sym}(D_v)$) and of partial bijections β_e for every $e \notin \mathscr{T}$ satisfying the following conditions:

- for every $e \in \mathcal{E}(\mathscr{H}) \setminus \mathcal{E}(\mathscr{T})$,
  - $\mathrm{dom}(\beta_e)$ is $\alpha_{\mathbf{s}(e)}(\iota_{e,\mathbf{s}}(G_e))$-stable;
  - $\mathrm{rng}(\beta_e)$ is $\alpha_{\mathbf{t}(e)}(\iota_{e,\mathbf{t}}(G_e))$-stable;
- for every $e \in \mathcal{E}(\mathscr{T})$, for every $g \in G_e$ and $x \in D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}$, one has

$$x \cdot \alpha_{\mathbf{s}(e)}(\iota_{e,\mathbf{s}}(g)) = x \cdot \alpha_{\mathbf{t}(e)}(\iota_{e,\mathbf{t}}(g));$$

- for every $e \in \mathcal{E}(\mathscr{H}) \smallsetminus \mathcal{E}(\mathscr{T})$, for every $g \in G_e$ and $x \in \mathrm{dom}(\beta_e) \cap D_{\mathbf{s}(e)} \cap \beta_e^{-1}\left(D_{\mathbf{t}(e)}\right)$, one has
$$x \cdot \alpha_{\mathbf{s}(e)}\left(\iota_{e,\mathbf{s}}(g)\right) \cdot \beta_e = x \cdot \beta_e \cdot \alpha_{\mathbf{t}(e)}\left(\iota_{e,\mathbf{t}}(g)\right);$$
- for every vertices $v, w \in \mathcal{V}(\mathscr{H})$, for every vertex $u \in [v, w]_{\mathscr{T}}$, one has
$$D_v \cap D_w \subseteq D_u;$$
- for every $e \in \mathcal{E}(\mathscr{H}) \smallsetminus \mathcal{E}(\mathscr{T})$ and $v \in \mathcal{V}(\mathscr{H})$, for every $u \in [v, \mathbf{s}(e)]_{\mathscr{T}}$, one has
$$D_v \cap \mathrm{dom}(\beta_e) \subseteq D_u.$$

To alleviate notations, given a $\mathscr{H}$-preaction α defined on a countable set X, we will simply denote by $x \cdot g$ the element $x \cdot \alpha_v(g)$ if $x \in D_v$, and by $x \cdot t_e$ the element $x \cdot \beta_e$ if $x \in \mathrm{dom}(\beta_e)$.

Informally, the $\mathscr{H}$-graph of a preaction α is the Schreier graph of α all of whose G_v-orbits are shrunk to vertices labeled by the corresponding G_v-actions for every vertex group G_v. As the isomorphism class of a transitive right action of a countable group on a countable set X is uniquely determined by the stabilizer of any point of X, these labels will be G_v-conjugacy classes of subgroups of G_v. For every vertex $x \cdot G_v$, $x' \cdot G_w$, we put an edge labeled $e \in \mathcal{E}(\mathscr{H})$ between $x \cdot G_v$ and $x' \cdot G_w$ iff $(\mathbf{s}(e), \mathbf{t}(e)) = (v, w)$, $x \in \mathrm{dom}(t_e)$ and

- either $e \in \mathscr{T}$ and $x \cdot G_v \cap x' \cdot G_w \neq \varnothing$;
- or $e \notin \mathscr{T}$ and $xt_e \cdot G_v \cap x' \cdot G_w \neq \varnothing$.

Because of the fact that vertex groups need not be commutative for the moment, we also add a label to the inferior and the superior half-edge in order to remember "where" do the two orbits $x \cdot G_v$ and $x' \cdot G_w$ (or $xt_e \cdot G_v$ and $x' \cdot G_w$) intersect.

We now give the formal definition:

Definition 3.2. Let α be a $\mathscr{H}$-preaction of G on a countable set X. One defines the $\mathscr{H}$-graph $\mathcal{G}$ of α as follows :

- its vertex set is the set of G_v-orbits for every $v \in \mathcal{V}(\mathscr{H})$:
$$\mathcal{V}(\mathcal{G}) = \bigsqcup_{v \in \mathcal{V}(\mathscr{H})} D_v / G_v;$$

- its edge set is $\mathcal{E}(\mathcal{G}) = \mathcal{E}^+(\mathcal{G}) \sqcup \mathcal{E}^-(\mathcal{G})$ where

  $$\mathcal{E}^+(\mathcal{G}) = \bigsqcup_{e\in\mathcal{E}^+(\mathscr{T})} \left(D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}\right)/\iota_{e,\mathbf{s}}(G_e) \bigsqcup \bigsqcup_{e\in\mathcal{E}^+(\mathscr{H})\smallsetminus\mathcal{E}(\mathscr{T})} \left(D_{\mathbf{s}(e)} \cap t_e^{-1}\left(D_{\mathbf{t}(e)}\right)\right)/\iota_{e,\mathbf{s}}(G_e)$$

  and

  $$\mathcal{E}^-(\mathcal{G}) = \bigsqcup_{e\in\mathcal{E}^+(\mathscr{T})} \left(D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}\right)/\iota_{e,\mathbf{t}}(G_e) \bigsqcup \bigsqcup_{e\in\mathcal{E}^+(\mathscr{H})\smallsetminus\mathcal{E}(\mathscr{T})} \left(D_{\mathbf{t}(e)} \cap t_e\left(D_{\mathbf{s}(e)}\right)\right)/\iota_{e,\mathbf{t}}(G_e)$$

  with

  - for every $e \in \mathcal{E}^+(\mathscr{T})$, for every $x \in D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}$:

    $$\mathbf{s}\left(x\iota_{e,\mathbf{s}}(G_e)\right) = xG_{\mathbf{s}(e)}$$

    and

    $$\mathbf{t}\left(x\iota_{e,\mathbf{t}}(G_e)\right) = xG_{\mathbf{t}(e)}$$

    Moreover

    $$\overline{x\iota_{e,\mathbf{s}}(G_e)} = x\iota_{e,\mathbf{t}}(G_e);$$

  - for every $e \in \mathcal{E}^+(\mathscr{H}) \smallsetminus \mathcal{E}(\mathscr{T})$, for every $x \in D_{\mathbf{s}(e)} \cap t_e^{-1}\left(D_{\mathbf{t}(e)}\right)$:

    $$\mathbf{s}\left(x\iota_{e,\mathbf{s}}(G_e)\right) = xG_{\mathbf{s}(e)}$$

    and

    $$\mathbf{t}\left(x\iota_{e,\mathbf{t}}(G_e)\right) = xt_eG_{\mathbf{t}(e)}$$

    Moreover

    $$\overline{x\iota_{e,\mathbf{s}}(G_e)} = xt_e\iota_{e,\mathbf{t}}(G_e);$$

- each vertex xG_v is labeled $([\mathrm{Stab}_{G_v}(x)]_{G_v}, v)$;

- for every vertex xG_v, we fix an identification between xG_v and the quotient set $\mathrm{Stab}_{G_v}(x)\backslash G_v$ (in an equivariant way). Each edge $(\mathrm{Stab}_{G_v}(x)g)\iota_{e,\mathbf{s}}(G_e)$ is labeled e and its inferior half-edge is labeled $(\mathrm{Stab}_{G_v}(x)g)\iota_{e,\mathbf{s}}(G_e)$; in particular (applying this last condition to $\overline{e}$), it is labeled $\overline{(\mathrm{Stab}_{G_v}(x)g)\iota_{e,\mathbf{s}}(G_e)}$ at its target.

Remark 3.3. In Item 3.2, the data of $[\mathrm{Stab}_{G_v}(x)]_{G_v}$ is equivalent to the data of the G_v-action $\mathrm{Stab}_{G_v}(x)\backslash G_v \curvearrowleft G_v$, or equivalently, of the G_v -action on xG_v.

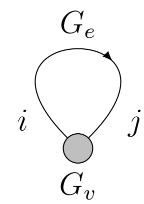

Figure 1: Graph of groups $\mathscr{H}$ defining an HNN-extension

Example 3.4. Let us consider the HNN-extension G of some group G_v over a group G_e defined by the two inclusions $i : G_e \hookrightarrow G_v$ and $j : G_e \hookrightarrow G_v$. It is the fundamental group of the graph of groups defined in Figure 1.

The group G inherits the following presentation

$$G \simeq \langle G_v, t_e \mid t_e^{-1} i(g) t_e = j(g) \rangle. \tag{3.5}$$

Let us consider a $\mathscr{H}$-preaction that consists of two G_v-orbits $x \cdot G_v$, $y \cdot G_v$, and such that t_e sends two points of $x \cdot G_v$ (say x, x') that lie in two different $i(G_e)$-orbits to two other points in $y \cdot G_v$ (say y, y') that lie in two different $j(G_e)$-orbits (see Figure 2).

The $\mathscr{H}$-graph of the above action consists of

- two vertices, that correspond to the two G_v-orbits;
- two edges between those vertices, that correspond to the $i(G_e), j(G_e)$-orbits.

This graph is represented in Figure 3.

Notice that the chosen identifications required in Item 3.2 send x (*resp.* y) to the coset $\mathrm{Stab}_{G_v}(x)1$ (*resp.* $\mathrm{Stab}_{G_v}(y)1$) of $\mathrm{Stab}_{G_v}(x)\backslash G_v$ (*resp.* $\mathrm{Stab}_{G_v}(y)\backslash G_v$).

Similarly, one defines the $\mathscr{H}$-graph of a subgroup H using the correpondence between subgroups and right actions:

Definition 3.6. The $\mathscr{H}$-graph of a subgroup H of G is the $\mathscr{H}$-graph of the right action of G on $H\backslash G$.

Remark 3.7. As observed in [Bon24] in the case of infinite cyclic vertex and edge groups, the data of the graph of groups of H is equivalent to its $\mathscr{H}$-graph: recall that the set of vertices of the Bass-Serre tree $\mathcal{T}$ associated to a graph of groups $\mathscr{H}$ of fundamental group G is $\bigsqcup_{v \in \mathcal{V}(\mathscr{H})} G/G_v$, and its set of edges is $\bigsqcup_{e \in \mathcal{E}(\mathscr{H})} G/G_e$. Thus, for any subgroup H of G, the set of vertices (*resp.* of edges) of $H\backslash\mathcal{T}$ is exactly

$$\bigsqcup_{v \in \mathcal{V}(\mathscr{H})} H\backslash(G/G_v) = \bigsqcup_{v \in \mathcal{V}(\mathscr{H})} (H\backslash G)/G_v,$$

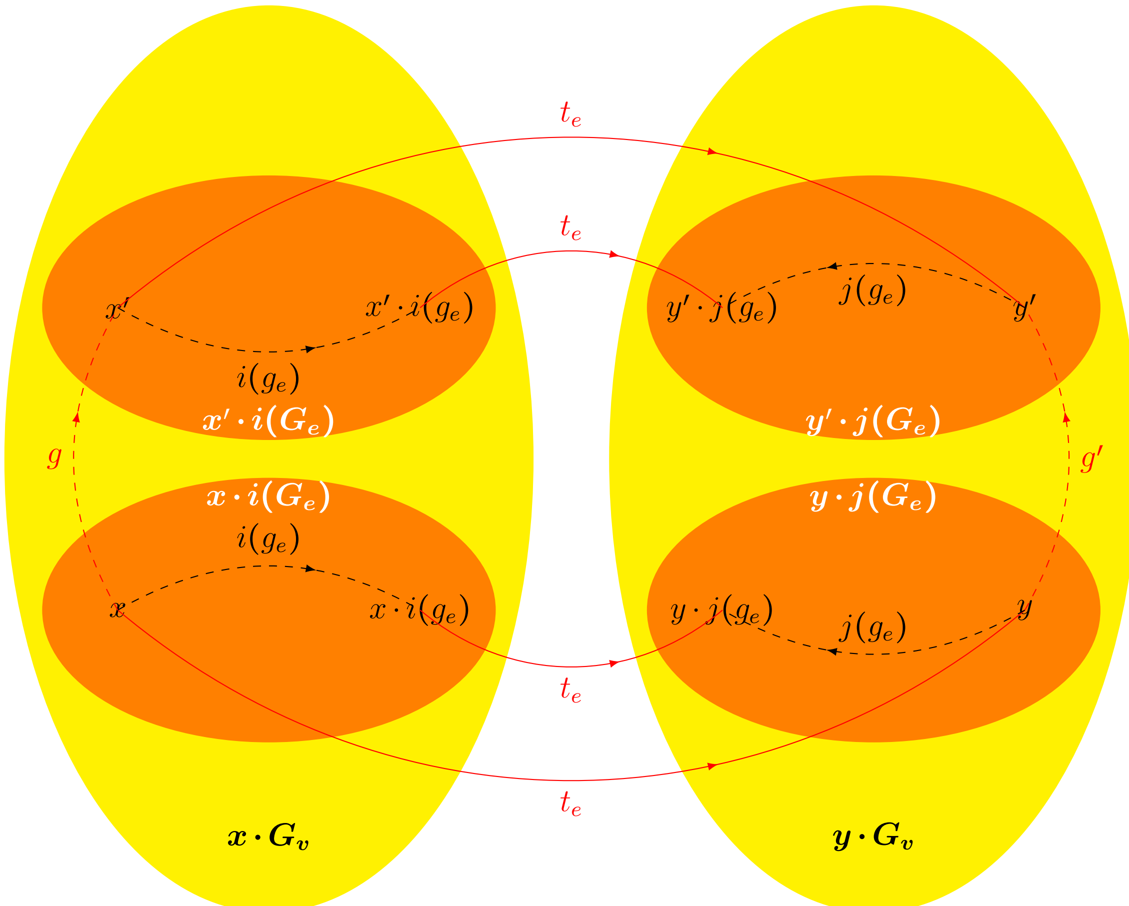

Figure 2: Schreier graph of a $\mathscr{H}$-preaction (here $g, g' \in G_v$ and $g_e \in G_e$)

(*resp.* $\bigsqcup_{e \in \mathcal{E}(\mathscr{H})} H\backslash(G/G_e) = \bigsqcup_{e \in \mathcal{E}(\mathscr{H})}(H\backslash G)/G_e)$), which is exactly the set of vertices (*resp.* of edges) of the $\mathscr{H}$-graph of H (taking the quotient $H\backslash G$ amount to taking the Schreier graph of H, then taking the disjoint union over the vertices of $\mathscr{H}$ amounts to "blow it up", and taking the quotient by every vertex (*resp.* edge) group amounts to shrinking the orbits of vertex groups (*resp.* edge groups), which amounts to constructing the vertices (*resp.* edges) of the $\mathscr{H}$-graph of H).

Now we want to define an abstract notion of $\mathscr{H}$-graph. To achieve this, we first prove the following lemma, which gives a combinatorial condition on the labels of the vertices of the $\mathscr{H}$-graph of a preaction.

Lemma 3.8. *Let $\mathcal{G}$ be the $\mathscr{H}$-graph of a $\mathscr{H}$-preaction α defined on a set X_0. For every $E \in \mathcal{E}(\mathcal{G})$, denoting by e the label of E, by $([\Lambda_0]_{G_{\mathbf{s}(e)}}, \mathbf{s}(e))$ (resp. $([\Lambda_1]_{G_{\mathbf{t}(e)}}, \mathbf{t}(e))$)*

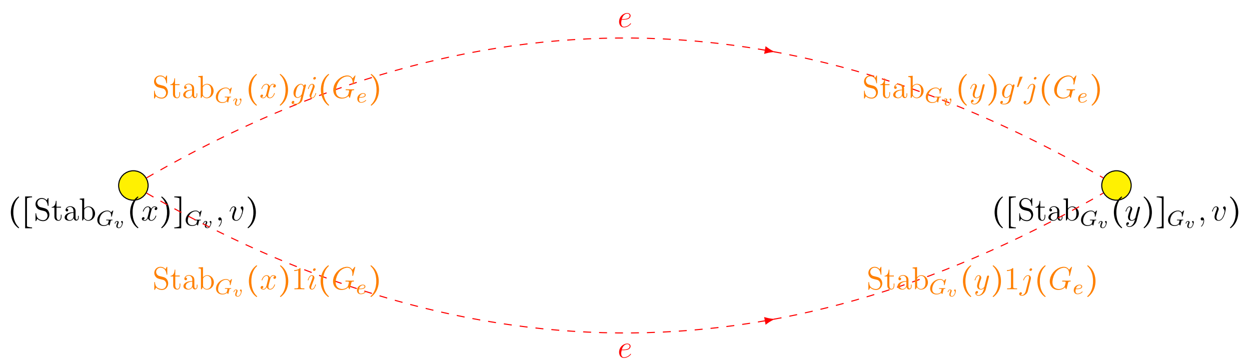

Figure 3: $\mathscr{H}$-graph of the $\mathscr{H}$-preaction represented in Figure 2

the label of $\mathbf{s}(E)$ *(resp.* $\mathbf{t}(E)$*), by* $\Lambda_0 g_0 \iota_{e,\mathbf{s}}(G_e)$ *(resp.* $\Lambda_1 g_1 \iota_{e,\mathbf{t}}(G_e)$*) the label of the inferior (resp. superior) half-edge of* E*, one has*

$$\left[\iota_{e,\mathbf{s}}^{-1}\left(g_0^{-1}\Lambda_0 g_0\right)\right]_{G_e} = \left[\iota_{e,\mathbf{t}}^{-1}\left(g_1^{-1}\Lambda_1 g_1\right)\right]_{G_e}.$$

Proof. For every $e \in \mathcal{E}(\mathscr{H})$, let us define $s_e = \begin{cases} t_e & \text{if } e \notin \mathcal{E}(\mathscr{T}) \\ id & \text{otherwise} \end{cases}$. By construction, there exist $x, y \in X_0$ and $g, g' \in G_e$ such that

- $\mathrm{Stab}_{G_{\mathbf{s}(e)}}(x) = \Lambda_0$;
- $\mathrm{Stab}_{G_{\mathbf{t}(e)}}(y) = \Lambda_1$;
- $x \cdot g_0 \iota_{e,\mathbf{s}}(g') s_e = y \cdot g_1 \iota_{e,\mathbf{t}}(g)$.

Hence, denoting by $X = x \cdot g_0 \iota_{e,\mathbf{s}}(g)$ and $Y = y \cdot g_1 \iota_{e,\mathbf{t}}(g')$, one has $Y = X s_e$, thus

$$\begin{aligned}
\mathrm{Stab}_{G_{\mathbf{t}(e)}}(Y) \cap \iota_{e,\mathbf{t}}(G_e) &= \mathrm{Stab}_{\iota_{e,\mathbf{t}}(G_e)}(Y) \\
&= \mathrm{Stab}_{s_e^{-1}\iota_{e,\mathbf{s}(G_e)}s_e}(X s_e) \\
&= s_e^{-1}\left(\mathrm{Stab}_{\iota_{e,\mathbf{s}(G_e)}}(X)\right) s_e \\
&= s_e^{-1}\left(\mathrm{Stab}_{G_{\mathbf{s}(e)}}(X) \cap \iota_{e,\mathbf{s}}(G_e)\right) s_e \\
&= \iota_{e,\mathbf{t}}\left(\iota_{e,\mathbf{s}}^{-1}\left(\mathrm{Stab}_{G_{\mathbf{s}(e)}}(X)\right)\right)
\end{aligned}$$

so, taking the preimage under $\iota_{e,\mathbf{t}}$,we get

$$\iota_{e,\mathbf{s}}^{-1}(\mathrm{Stab}_{G_{\mathbf{s}(e)}}(X)) = \iota_{e,\mathbf{t}}^{-1}(\mathrm{Stab}_{G_{\mathbf{t}(e)}}(Y))$$

which amounts to saying that

$$g^{-1}\iota_{e,\mathbf{s}}^{-1}\left(g_0^{-1}\Lambda_0 g_0\right)g = g'^{-1}\iota_{e,\mathbf{t}}^{-1}\left(g_1^{-1}\Lambda_1 g_1\right)g'$$

which proves the statement. □

Using this lemma, one can extend the notion of a $\mathscr{H}$-graph as follows:

Definition 3.9. A $\mathscr{H}$-graph $\mathcal{G}$ is a labeled graph satisfying the following conditions:

- every vertex is labeled $([\Lambda]_{G_v}, v)$ for some $v \in \mathcal{V}(\mathscr{H})$ and some subgroup Λ of G_v;
- every edge $E \in \mathcal{E}(\mathcal{G})$ is labeled by an edge $e \in \mathcal{E}(\mathscr{H})$ such that $\mathbf{s}(E)$ is labeled $(C, \mathbf{s}(e))$ (for some $G_{\mathbf{s}(e)}$-conjugacy class C of subgroups of $G_{\mathbf{s}(e)}$) and $\mathbf{t}(E)$ is labeled $(C', \mathbf{t}(e))$ (for some $G_{\mathbf{t}(e)}$-conjugacy class C' of subgroups of $G_{\mathbf{t}(e)}$);
- the inferior half-edge of an edge labeled e with source labeled $([\Lambda]_{G_v}, v)$ is labeled by an element of $\Lambda\backslash G_v/\iota_{e,\mathbf{s}}(G_e)$;
- every edge labeled e whose inferior half-edge is labeled $\Lambda_0 g_0 \iota_{e,\mathbf{s}}(G_e)$ (with $\Lambda_0 \leq G_{\mathbf{s}(e)}$) and whose superior half-edge is labeled $\Lambda_1 g_1 \iota_{e,\mathbf{t}}(G_e)$ (with $\Lambda_1 \leq G_{\mathbf{t}(e)}$) satisfies
$$\left[\iota_{e,\mathbf{s}}^{-1}\left(g_0^{-1}\Lambda_0 g_0\right)\right]_{G_e} = \left[\iota_{e,\mathbf{t}}^{-1}\left(g_1^{-1}\Lambda_1 g_1\right)\right]_{G_e};$$
- inferior (*resp.* superior) half-edges of different edges labeled e (for some $e \in \mathcal{E}(\mathscr{H})$) sharing the same source (*resp.* target) can't share the same label.

Notice that a $\mathscr{H}$-graph is the $\mathscr{H}$-graph of a subgroup $H \leq G$ if and only if every vertex labeled $([\Lambda]_{G_v}, v)$ has exactly $|\Lambda\backslash G_v/\iota_{e,\mathbf{s}}(G_e)|$ outgoing edges labeled e for every edge $e \in \mathcal{E}(\mathscr{H})$ satisfying $\mathbf{s}(e) = v$ (we call such $\mathcal{G}$ a **saturated** $\mathscr{H}$-graph). In the formalism of Bass, who introduced the right notion of coverings in the setting of graphs of groups in [Bas93], a $\mathscr{H}$-graph $\mathcal{G}$ is the base space of an *immersion* $\mathcal{G} \to \mathscr{H}$ of the graph of groups $\mathscr{H}$. This immersion is a *covering* if and only if $\mathcal{G}$ is saturated, that is to say, $\mathcal{G}$ is the $\mathscr{H}$-graph of a subgroup of G.

The three following lemmas (3.11, 3.10 and 3.14) give rise to a useful tool to approximate subgroups of G. We show that, under some conditions, an approximation of the $\mathscr{H}$-graph of a subgroup gives rise to an approximation of the subgroup itself. The proofs of these are very similar to the ones given in the case of GBS groups of rank 1 in [Bon24] (Lemma 3.3, Lemma 3.4 and Lemma 3.5). For the convenience of the reader, we adapt the main ingredients of the proofs to our wider setting.

Lemma 3.10. *Let $\mathcal{F}$ be a finite $\mathscr{H}$-graph. There exists a $\mathscr{H}$-preaction whose $\mathscr{H}$-graph is $\mathcal{F}$.*

Lemma 3.11. *Let $(\alpha_i)_{i\in\mathbb{N}}$ be a collection of $\mathscr{H}$-preactions whose $\mathscr{H}$-graphs $(\mathcal{G}_i)_{i\in\mathbb{N}}$ are contained in a saturated $\mathscr{H}$-graph $\mathcal{G}$ as pairwise disjoint subgraphs such that the quotient $\mathcal{G}/(\bigsqcup_{i\in\mathbb{N}}\mathcal{G}_i)$ is a tree. There exists a G-action α whose $\mathscr{H}$-graph is $\mathcal{G}$ and that extends α_i for every $i\in\mathbb{N}$.*

The proofs of Lemma 3.11 and Lemma 3.10 rely on straightforward inductions based on the following proposition:

Proposition 3.12. *Let α_0 be a $\mathscr{H}$-preaction defined on a countable set X_0 whose $\mathscr{H}$-graph $\mathcal{G}_0$ is contained in a $\mathscr{H}$-graph $\mathcal{G}$ such that:*

- $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}_0)$;
- $\mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{G}_0)\sqcup\{E\}$ *for some edge E such that* $\mathbf{s}(E),\mathbf{t}(E)\in\mathcal{V}(\mathcal{G}_0)$.

Then, there exists a $\mathscr{H}$-preaction α whose $\mathscr{H}$-graph is $\mathcal{G}$. Moreover, if $\mathbf{s}(E)$ and $\mathbf{t}(E)$ belong to two different connected components of $\mathcal{G}_0$ (that is two say, $\alpha_0 = \alpha_1\sqcup\alpha_2$ for some subpreactions α_1 and α_2, and $\mathbf{s}(E)$ (resp. $\mathbf{t}(E)$) corresponds to a vertex orbit for α_1 (resp. $\mathbf{t}(E)$), the constructed preaction α extends both α_1 and α_2.

Proof. We adapt Constructions A and B defined in [Bon24, Section 3]. Let e be the label of E and let us denote by $(V_1,V_2) \coloneqq (\mathbf{s}(E),\mathbf{t}(E))$, and by $(v_1,v_2)\coloneqq(\mathbf{s}(e),\mathbf{t}(e))$. Let $([\Lambda_i]_{G_{v_i}}, v_i)$ be the label of V_i (for $i\in\{1,2\}$) and let $\Lambda_1 g_1\iota_{e,\mathbf{s}}(G_e)$ and $\Lambda_2 g_2\iota_{e,\mathbf{t}}(G_e)$ be the label of the inferior half-edge and of the superior half-edge of E, respectively. One has $[\iota_{e,\mathbf{s}}^{-1}(g_1^{-1}\Lambda_1 g_1)]_{G_e} = [\iota_{e,\mathbf{t}}^{-1}(g_2^{-1}\Lambda_2 g_2)]_{G_e}$, *i.e.* there exists $h\in G_e$ such that

$$\iota_{e,\mathbf{s}}^{-1}(g_1^{-1}\Lambda_1 g_1) = h^{-1}\iota_{e,\mathbf{t}}^{-1}(g_2^{-1}\Lambda_2 g_2)h. \tag{3.13}$$

We distinguish two cases:

Construction A : If $e\notin\mathcal{E}(\mathscr{T})$, then there exist $x_1,x_2\in X_0$ such that:

- $x_i\in\operatorname{dom}(G_{v_i})$ for $i\in\{1,2\}$;
- $\operatorname{Stab}_{G_{v_i}}(x_i) = \Lambda_i$ for $i\in\{1,2\}$;
- $x_1\cdot g_1\notin\operatorname{dom}(t_e)$ and $x_2\cdot g_2\notin\operatorname{rng}(t_e)$.

We extend t_e and t_e^{-1} on a subset of $X\coloneqq X_0$ by defining, for every $g\in G_e$:

$$(x_1\cdot g_1(\iota_{e,\mathbf{s}}(g)))\cdot t_e = x_2\cdot g_2\iota_{e,\mathbf{t}}(hgh^{-1}),$$

which is well-defined by Equation 3.13. The resulting $\mathscr{H}$-preaction is suitable.

Construction B : If $e \in \mathcal{E}(\mathscr{T})$, then there exist $x_1, x_2 \in X_0$ such that:

- $x_i \in \operatorname{dom}(G_{v_i})$ for $i \in \{1,2\}$;
- $\operatorname{Stab}_{G_{v_i}}(x_i) = \Lambda_i$ for $i \in \{1,2\}$;
- $x_1 \cdot g_1 \notin \operatorname{dom}(G_{v_2})$ and $x_2 \cdot g_2 \notin \operatorname{dom}(G_{v_1})$.

We let

$$X \coloneqq X_0 / \big(x_1 \cdot g_1 \iota_{e,\mathbf{s}}(g) \sim x_2 \cdot g_2 \iota_{e,\mathbf{t}}(hgh^{-1}) \ \forall g \in G_e \big)$$

and we let α be the $\mathscr{H}$-preaction induced by α on X. As previously, it is well-defined by Equation 3.13. By construction, the $\mathscr{H}$-graph of α is $\mathcal{G}$. □

Finally we explain how to saturate a $\mathscr{H}$-graph in our new setting.

Lemma 3.14. *For every (non-saturated) $\mathscr{H}$-graph $\mathcal{G}$, there exists a saturated $\mathscr{H}$-graph $\tilde{\mathcal{G}}$ that contains $\mathcal{G}$ and such that the quotient $\tilde{\mathcal{G}}/\mathcal{G}$ is an infinite forest.*

Proof. We argue by induction using the following construction: if V is a non-saturated vertex labeled $([\Lambda_0]_{G_v}, v)$, whose non-saturation is witnessed by

- an edge $e \in \mathcal{E}(\mathscr{H})$ with source v;
- an element $g_0 \in G_v$ such that there is no inferior half-edge labeled $\Lambda_0 g_0 \iota_{e,\mathbf{s}}(G_e)$ with source V

then, denoting by $w = \mathbf{t}(e)$, one defines

$$\Lambda_1 = \iota_{e,\mathbf{t}}(\iota_{e,\mathbf{s}}^{-1}(g_0^{-1}\Lambda_0 g_0))$$

and one builds a new vertex W labeled $([\Lambda_1]_{G_w}, w)$ and a new edge E labeled e with source V and target W such that

- the inferior half-edge $(E, \mathbf{s})$ is labeled $\Lambda_0 g_0 \iota_{e,\mathbf{s}}(G_e)$;
- the superior half-edge $(E, \mathbf{t})$ is labeled $\Lambda_1 \iota_{e,\mathbf{t}}(G_e)$.

□

3.2 Generalized Baumslag-Solitar groups

In this section, we will apply the tools we introduced in the previous section to GBS groups.

A GBS group of rank d is the fundamental group of a finite graph of groups whose vertex and edge stabilizers are isomorphic to $\mathbb{Z}^d$. As any injective morphism $\mathbb{Z}^d \to \mathbb{Z}^d$ is represented by an element of $M_d(\mathbb{Z}) \cap GL_d(\mathbb{Q})$, a GBS group can be represented by an oriented graph $\mathscr{H}$ endowed with a function which associates an integer matrix whose determinant is non-zero to each half-edge:

$$\begin{array}{rcccc} \mathrm{M} & : & \mathcal{E}(\mathscr{H}) \times \{\mathbf{s}, \mathbf{t}\} & \to & M_d(\mathbb{Z}) \cap GL_d(\mathbb{Q}) \\ & & (e, \mathbf{u}) & \mapsto & \mathrm{M}_{e,\mathbf{u}} \end{array}$$

and that satisfies $\mathrm{M}_{\overline{e},\mathbf{t}} = \mathrm{M}_{e,\mathbf{s}}$ for every $e \in \mathcal{E}(\mathscr{H})$. Up to shrinking some edges, we can assume that the graph $\mathscr{H}$ is **reduced**, that is to say, the only edges $e \in \mathcal{V}(\mathscr{H})$ one of whose labels is in $GL_d(\mathbb{Z})$ are loops.

Given a GBS group G defined by a graph of groups $\mathscr{H}$, let us denote by $\mathcal{T}$ its Bass-Serre tree and let us define the **modular homomorphism** as follows (*cf.* [LdGZS25]): fix a vertex $v \in \mathcal{V}(\mathcal{T})$. Observe that for any $g \in G$, the group $G_{gv} \cap G_v$ has finite index in G_v, hence belongs to the abstract commensurator of G_v. Define $\Delta_G^{(v)}(g)$ as the equivalence class of the morphism $\begin{array}{ccc} G_v \cap G_{g^{-1}v} & \to & G_v \cap G_{gv} \\ h & \mapsto & ghg^{-1} \end{array}$. As G_v is isomorphic to $\mathbb{Z}^d$, the morphism $\Delta_G^{(v)}$ can be identified with a morphism $\Delta_G^{(v)} : G \to GL_d(\mathbb{Q})$. The definition of the modular homomorphism does not depend on the choice of the vertex v up to conjugation by an element of $GL_d(\mathbb{Q})$. Practically, the image of the modular homomorphism (based at some vertex v) is generated by the matrices $\mathrm{B}_n\mathrm{A}_n^{-1}\dots\mathrm{B}_1\mathrm{A}_1^{-1}$ for every edge path $e_1, ..., e_n$ labeled $(\mathrm{A}_1, \mathrm{B}_1)$, ... ,$(\mathrm{A}_n, \mathrm{B}_n)$ and based at v. In particular, the modular homomorphism of a GBS group defined by a tree of groups is trivial. A GBS group G of rank d is **unimodular** if $\mathrm{Im}(\det \circ \Delta_G) \subseteq \{1, -1\}$.

Example 3.15. Let us consider the GBS of rank 2 defined in Figure 4.

Let us chose the edge f as a spanning tree in the graph represented in Figure 4. Then, the presentation of G associated to this choice is the following:

$$\begin{aligned} G \simeq \langle x_v, y_v, x_w, y_w, t_e \mid & [x_v, y_v] = [x_w, y_w] = 1, \\ & x_v^7 y_v^{-3} = x_w^{-4} y_w^3, x_v^{-1} y_v^{-3} = x_w y_w^{-9}, \\ & t_e^{-1}(x_v y_v^{-2}) t_e = x_w^{-1} y_w^4, t_e^{-1} x_v^5 t_e = x_w^8 y_w^5 \rangle . \end{aligned}$$

The modular homomorphism is trivial on the vertex generators x_v, y_v, x_w, y_w and

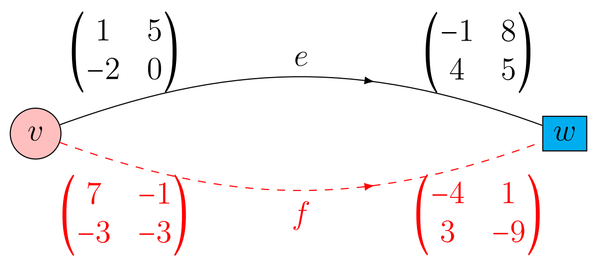

Figure 4: A graph of GBS

sends t_e to the product $\begin{pmatrix}7 & -1\\ -3 & -3\end{pmatrix} \cdot \begin{pmatrix}-4 & 1\\ 3 & 9\end{pmatrix}^{-1} \cdot \begin{pmatrix}-1 & 8\\ 4 & 5\end{pmatrix} \cdot \begin{pmatrix}1 & 5\\ -2 & 0\end{pmatrix}^{-1}$:

$$\Delta_v : \begin{array}{ccc} G & \to & GL_2(\mathbb{Q}) \\ x_v, y_v, x_w, y_w & \mapsto & I_2 \\ t_e & \mapsto & \frac{-1}{390}\begin{pmatrix}1026 & 903\\ -138 & -459\end{pmatrix} \end{array}$$

In the case where the vertex stabilizers are abelian, the definition of a $\mathscr{H}$-graph simplifies as follows. A $\mathscr{H}$-graph $\mathcal{G}$ is then a labeled graph satisfying the following conditions:

1. every vertex is labeled (Λ, v) for some $v \in \mathcal{V}(\mathscr{H})$ and some subgroup Λ of G_v;

2. every edge $E \in \mathcal{E}(\mathcal{G})$ is labeled e for some $e \in \mathcal{E}(\mathscr{H})$ such that $\mathbf{s}(E)$ is labeled $(\Lambda_0, \mathbf{s}(e))$ (for some $\Lambda_0 \le G_{\mathbf{s}(e)}$) and $\mathbf{t}(E)$ is labeled $(\Lambda_1, \mathbf{t}(e))$ (for some $\Lambda_1 \le G_{\mathbf{t}(e)}$);

3. every edge labeled e whose source (*resp.* target) is labeled $(\Lambda_0, \mathbf{s}(e))$ (*resp.* $(\Lambda_1, \mathbf{t}(e))$) satisfies
$$\iota_{e,\mathbf{s}}^{-1}(\Lambda_0) = \iota_{e,\mathbf{t}}^{-1}(\Lambda_1)\,;$$

4. every vertex labeled (Λ, v) has at most $|\Lambda \backslash G_v / \iota_{e,\mathbf{s}}(G_e)|$ outgoing edges labeled e (with $\mathbf{s}(e) = v$) and at most $|\Lambda \backslash G_v / \iota_{e,\mathbf{t}}(G_e)|$ incoming edges labeled e (with $\mathbf{t}(e) = v$).

Observe that a $\mathscr{H}$-graph is the $\mathscr{H}$-graph of a subgroup iff equality holds for every vertex and edge in the last item.

Hence for a GBS group of rank d the definition of a $\mathscr{H}$-graph becomes the following:

Definition 3.16. Let $d \geq 1$ and let $\mathscr{H}$ be a finite graph of groups all of whose vertex and edge groups are isomorphic to $\mathbb{Z}^d$. A $\mathscr{H}$-graph is a labeled graph that satisfies the three following conditions:

1. every vertex is labeled (Λ, v) for some $v \in \mathcal{V}(\mathscr{H})$ and some subgroup Λ of $\mathbb{Z}^d$;

2. every edge $E \in \mathcal{E}(\mathcal{G})$ is labeled e for some $e \in \mathcal{E}(\mathscr{H})$ such that $\mathbf{s}(E)$ is labeled $(\Lambda_0, \mathbf{s}(e))$ (for some $\Lambda_0 \leq G_{\mathbf{s}(e)}$) and $\mathbf{t}(E)$ is labeled $(\Lambda_1, \mathbf{t}(e))$ (for some $\Lambda_1 \leq G_{\mathbf{t}(e)}$);

3. **Transfer Equation** every edge labeled e whose source (*resp.* target) is labeled $(\Lambda_0, \mathbf{s}(e))$ (*resp.* $(\Lambda_1, \mathbf{t}(e))$) satisfies
$$\left(\mathrm{M}_{e,\mathbf{s}}^{-1}\Lambda_0\right) \cap \mathbb{Z}^d = \left(\mathrm{M}_{e,\mathbf{t}}^{-1}\Lambda_1\right) \cap \mathbb{Z}^d;$$

4. every vertex labeled (Λ, v) has at most $|\mathbb{Z}^d / \langle \Lambda, \mathrm{M}_{e,\mathbf{s}}\mathbb{Z}^d \rangle|$ incident edges labeled e (with $\mathbf{s}(e) = v$) and at most $|\mathbb{Z}^d / \langle \Lambda, \mathrm{M}_{e,\mathbf{t}}\mathbb{Z}^d \rangle|$ incident edges labeled e (with $\mathbf{t}(e) = v$);

It is saturated iff equality holds for vertex and edge in the last item.

Example 3.17. Let $\mathrm{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$. Let us defined the GBS group G_0 as the fundamental group $\pi_1(\mathscr{H}_0)$ of following graph of groups defined in Figure 5.

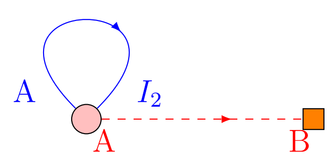

Figure 5: The graph of groups $\mathscr{H}_0$.

The labeled graph represented in Figure 6 is a (non-saturated) $\mathscr{H}_0$-graph.

We will focus on non-amenable GBS groups. They are characterized by the following proposition:

Proposition 3.18. *Let G be a GBS group defined by a reduced graph of groups $\mathscr{H}$. Then G is amenable iff $\mathscr{H}$ is a single loop one of whose labels is in $GL_2(\mathbb{Z})$, or a single edge e with $\mathbf{s}(e) \neq \mathbf{t}(e)$ both of whose labels have determinant ± 2.*

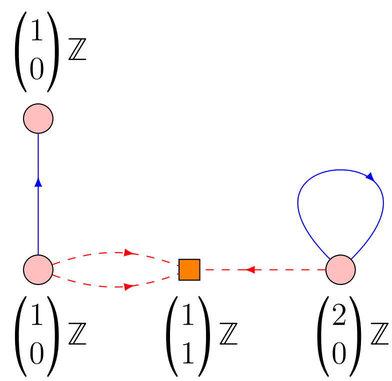

Figure 6: An example of a $\mathscr{H}_0$-graph.

Proof. Let us assume that $\mathscr{H}$ consists of a single loop e labeled (A, B) with $\mathrm{B} \in GL_d(\mathbb{Z})$. Denoting by $\mathrm{M} := \mathrm{AB}^{-1}$, the group G is isomorphic to $\mathbb{Z}[\mathrm{M}, \mathrm{M}^{-1}]\mathbb{Z}^d \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{Z}[\mathrm{M}, \mathrm{M}^{-1}]\mathbb{Z}^d$ multiplication by M. As an extension of an abelian group by an abelian group, G is amenable.

Now let us assume that $\mathscr{H}$ consists of an edge e which is not a loop such that the labels A and B of e have determinant ± 2. The Bass-Serre tree $\mathcal{T}$ of $\mathscr{H}$ is a bi-infinite line on which G acts with kernel N isomorphic to $\mathbb{Z}^d$. The action of G/N on $\mathcal{T}$ has also a single orbit of edges, the stabilizer of any edge is trivial, and, as A and B have determinant ± 2, the stabilizer of any vertex is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus, Bass-Serre theory tells us that G/N is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, which is virtually isomorphic to $\mathbb{Z}$. Hence, N and G/N are amenable, so G is amenable.

Conversely, let us assume that $\mathscr{H}$ is neither a single loop one of whose labels is in $GL_2(\mathbb{Z})$, nor a single edge e with $\mathbf{s}(e) \neq \mathbf{t}(e)$ both of whose labels have determinant ± 2. Then the action of G on its Bass-Serre tree $\mathcal{T}$ is of general type, thus G contains a free group on two generators. In particular, G is non-amenable. □

4 An equivalence relation on $\mathrm{Sub}(\mathbb{Z}^d)$

In this section, we assume that G is a non-amenable GBS group which is *not* the fundamental group of a graph of groups defined by a single vertex and a collection of loops labeled by invertible integer matrices. In other words, G is neither amenable nor isomorphic to a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$.

First, we introduce an equivalence relation that will give the decomposition of Theorem 1.2.

Definition 4.1. Let Λ_0 and Λ_1 be two subgroups of $\mathbb{Z}^d$. We say that Λ_0 and Λ_1

are $\mathscr{H}$-equivalent with respect to a vertex $v \in \mathcal{V}(\mathscr{H})$ (denoted w.r.t. v) if there exists a connected $\mathscr{H}$-graph that contains two vertices labeled (Λ_0, v) and (Λ_1, v), respectively.

Lemma 4.2. *Suppose that there exists a $\mathscr{H}$-path $E_1, ..., E_r$ that connects two vertices labeled (Λ_0, v) and (Λ_1, w), respectively. Then, for every edges $e, f \in \mathcal{E}(\mathscr{H})$ satisfying $\mathbf{s}(e) = v$ and $\mathbf{t}(f) = w$, there exists a $\mathscr{H}$-path of type $e, ..., e_1, ..., e_r, ..., f$ that connects two vertices labeled (Λ_0, v) and (Λ_1, w) and that contains $E_1, ..., E_r$ as a subpath.*

Proof. Let $E_1, ..., E_r$ be a reduced $\mathscr{H}$-path labeled $e_1, ..., e_r$ (with $e_i \in \mathcal{E}(\mathscr{H})$ for every $i \in [\![1, r]\!]$) such that

- $\mathbf{s}(E_1)$ is labeled (Λ_0, v);
- $\mathbf{t}(E_r)$ is labeled (Λ_1, w).

We first prove that there exists a reduced $\mathscr{H}$-path labeled $e, ..., e_1, ..., e_r$ that connects a vertex labeled (Λ_0, v) to a vertex labeled (Λ_1, w), and that contains $E_1, ..., E_r$ as a subpath. If $e = e_1$, then the edge path $E_1, ..., E_r$ is suitable.

Otherwise, let us denote by $(\mathrm{A}, \mathrm{B}) = (\mathrm{M}_{e,\mathbf{s}}, \mathrm{M}_{e,\mathbf{t}})$ and $v' = \mathbf{t}(e)$. Let us define

$$\Lambda_0' = \mathrm{B}(\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d).$$

By construction, (Λ_0, v) and (Λ_0', v') satisfy the Transfer Equation 3

$$\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d = \mathrm{B}^{-1}\Lambda_0' \cap \mathbb{Z}^d.$$

Hence there exists a $\mathscr{H}$-graph which consist of a single edge E_0' labeled e connecting a vertex V_0 labeled (Λ_0, v) to a vertex V_1 labeled (Λ_0', v').

Case 1: Let us first assume that $|\det(\mathrm{B})| \geq 2$. As $|\det(B)| \geq 2$ and $\Lambda_0' \subseteq \mathrm{B}\mathbb{Z}^d$, one has

$$\left|\mathbb{Z}^d / \langle \mathrm{B}\mathbb{Z}^d, \Lambda_0' \rangle\right| = |\det(\mathrm{B})| \geq 2.$$

Hence the labeled graph which consists of

- the edge E_0';
- an edge E_1' labeled $\overline{e}$ connecting V_1 to a vertex V_2 labeled (Λ_0, v)

is a $\mathscr{H}$-graph of type $e, \overline{e}$ that connects a vertex labeled (Λ_0, v) to a vertex labeled (Λ_0, v). Finally, as $e \neq e_1$ by assumption the labeled graph obtained by the concatenation of the $\mathscr{H}$-paths E_0', E_1' and $E_1, ..., E_r$ is a $\mathscr{H}$-graph of type $e, \overline{e}, e_1, ..., e_r$ that connects a vertex labeled (Λ_0, v) to a vertex labeled (Λ_1, w) as required.

Case 2 : Otherwise, $|\det(\mathrm{B})| = 1$. In particular, the graph $\mathscr{H}$ being reduced, e is a loop. We distinguish two subcases:

Subcase 2.1: There exists an edge $f \neq \overline{e}$ such that $|\det(\mathrm{M}_{f,\mathbf{t}})| \geq 2$. Hence we can apply Case 1 to obtain a $\mathscr{H}$-path E'_1, E'_2 of type $f, \overline{f}$ that connects a vertex labeled (Λ'_0, v) to a vertex labeled (Λ'_0, v). As $f \neq \overline{e}$ and (Λ_0, v) and (Λ'_0, v') satisfy the Transfer Equation 3, the labeled graph which consists of

- the concatenation of the $\mathscr{H}$-path E'_0 with the $\mathscr{H}$-path E'_1, E'_2;
- an edge E'_3 labeled $\overline{e}$ with source $\mathbf{t}(E'_2)$ and target a new vertex labeled (Λ_0, v)

is a $\mathscr{H}$-graph. As $e \neq e_1$, we can concatenate the $\mathscr{H}$-path E'_0, E'_1, E'_2, E'_3 and the $\mathscr{H}$-path $E_1, ..., E_r$ to obtain a $\mathscr{H}$-path labeled $e, f, \overline{f}, \overline{e}, e_1, ..., e_r$ that connects a vertex labeled (Λ_0) to a vertex labeled (Λ_1, w).

Subcase 2.2: Otherwise, $\mathscr{H}$ consists of a collection of at least two loops based at a single vertex such that every label (except possibly A) is in $GL_d(\mathbb{Q})$. As G is not a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$, one has necessarily $|\det(A)| \geq 2$. Let $f \in \mathscr{H} \setminus \{e, \overline{e}\}$. Subcase 2.1 delivers a $\mathscr{H}$-path E'_1, E'_2, E'_3, E'_4 of type $f, \overline{e}, e, \overline{f}$ that connects a vertex labeled (Λ'_0, v) to a vertex labeled (Λ'_0, v). As $f \neq \overline{e}$, the $\mathscr{H}$-graph which consists of

- the concatenation of the $\mathscr{H}$-path E'_0 with the $\mathscr{H}$-graph E'_1, E'_2, E'_3, E'_4;
- an edge E'_5 of type $\overline{e}$ that connects $\mathbf{t}(E'_4)$ to a new vertex labeled (Λ_0, v)

is a $\mathscr{H}$-path. As $e \neq e_1$, we can concatenate the $\mathscr{H}$-path $E'_0, E'_1, E'_2, E'_3, E'_4, E'_5$ and the $\mathscr{H}$-path $E_1, ..., E_r$ to obtain a $\mathscr{H}$-path labeled $e, f, \overline{e}, e, \overline{f}, \overline{e}, e_1, ..., e_r$ that connects a vertex labeled (Λ_0) to a vertex labeled (Λ_1, w).

Hence we proved that there exists a $\mathscr{H}$-path labeled $e, ..., e_1, ..., e_r$ that connects a vertex labeled (Λ_0, v) to a vertex labeled (Λ_1, w). Using this result on the reverse path (that connects a vertex labeled (Λ_1, w) to a vertex labeled (Λ_0, v)) leads to a $\mathscr{H}$-path of type $e, ..., e_1, ..., e_r, ..., f$ that connects a vertex labeled (Λ_0, v) to a vertex labeled (Λ_1, w), which leads to the conclusion.

□

Corollary 4.3. *$\mathscr{H}$-equivalence with respect to a prescribed vertex v is an equivalence relation.*

More precisely, the following holds: if $E_1, ..., E_r$ (resp. $F_1, ..., F_s$) is a $\mathscr{H}$-path that connects a vertex labeled (Λ_0, v) (resp. (Λ_1, v)) to a vertex labeled (Λ_1, v) (resp. (Λ_2, v)), then there exists a $\mathscr{H}$-path that contains $E_1, ..., E_{r-1}$ and $F_2, ..., F_s$ as disjoint (labeled) subpaths.

Proof. Let $\Lambda_0, \Lambda_1, \Lambda_2 \leq \mathbb{Z}^d$.

As the empty path is a $\mathscr{H}$-path that connects any vertex labeled (Λ_0, v) to itself, Λ_0 is $\mathscr{H}$-equivalent to itself (w.r.t. v).

If Λ_0 is $\mathscr{H}$-equivalent to Λ_1 w.r.t. v, then there exists a reduced $\mathscr{H}$-path $E_1, ..., E_r$ with source labeled (Λ_0, v) and target labeled (Λ_1, v). The reversed $\mathscr{H}$-path $\overline{E_r}, ..., \overline{E_1}$ has its source labeled (Λ_1, v) and its target labeled (Λ_0, v). Thus, Λ_1 is $\mathscr{H}$-equivalent to Λ_0 w.r.t. v.

Let us assume that Λ_0 is $\mathscr{H}$-equivalent to Λ_1 w.r.t. v and that Λ_1 is $\mathscr{H}$-equivalent to Λ_2 w.r.t. v Let $E_1, ..., E_r$ be a reduced $\mathscr{H}$-path of type $e_1, ..., e_r$ with source labeled (Λ_0, v) and target labeled (Λ_1, v) and let $F_1, ..., F_s$ be a reduced $\mathscr{H}$-path of type $f_1, ..., f_s$ with source labeled (Λ_1, v) and target labeled (Λ_2, v).

Case 1: Let us assume that $f_1 \neq \overline{e_r}$ Then, the concatenation of of these $\mathscr{H}$-paths delivers a $\mathscr{H}$-path with source labeled (Λ_0, v) and target labeled (Λ_2, v).

Case 2: Otherwise, let us denote by $e = e_r = \overline{f_1}$**.**

Subcase 2.1: First, we assume that there exists an edge $g \neq e_r$ **such that** $\mathbf{t}(g) = v$**.** By Lemma 4.2, there exists a $\mathscr{H}$-path $E_1', ..., E_t'$ of type $e_1, ..., e_r, ..., g$ that contains $E_1, ..., E_r$ and that connects a vertex labeled (Λ_0, v) to a vertex labeled (Λ_1, v). As $g \neq \overline{f_1}$, the concatenation of the $\mathscr{H}$-paths $E_1', ..., E_t'$ and $F_1, ..., F_s$ delivers a $\mathscr{H}$-path with source labeled (Λ_0, v) and target labeled (Λ_2, v).

Subcase 2.2: Now we assume that e **is the unique edge with target** v**.** Notice that in this case, e can't be a loop. If $r = 0$ or $s = 0$, we're done. Hence we assume that $r, s \geq 1$. Let $u = \mathbf{s}(e)$ and $(\mathrm{A}, \mathrm{B}) \coloneqq (\mathrm{M}_{e,\mathbf{s}}, M_{e,\mathbf{t}})$. Let $\left(\widetilde{\Lambda_1}, u\right)$ and $\left(\widetilde{\Lambda_2}, u\right)$ be the labels of $\mathbf{s}(E_r)$ and $\mathbf{t}(F_1)$, respectively. By the Transfer Equation 3 we have

$$\begin{aligned} \mathrm{A}^{-1}\widetilde{\Lambda_1} \cap \mathbb{Z}^d &= \mathrm{B}^{-1}\Lambda_1 \cap \mathbb{Z}^d \\ &= \mathrm{A}^{-1}\widetilde{\Lambda_2} \cap \mathbb{Z}^d. \end{aligned}$$

Let us define

$$\Lambda_1' = \mathrm{B}(\mathrm{A}^{-1}\widetilde{\Lambda_1} \cap \mathbb{Z}^d).$$

As e is not a loop and $\mathscr{H}$ is reduced, one has $|\det(B)| \geq 2$. Hence the labeled graph which consists of

- an edge E_0' labeled e with source labeled $\left(\widetilde{\Lambda_1}, u\right)$ and target V labeled (Λ_1', v);
- an edge E_1' labeled e with source V and target a vertex labeled $\left(\widetilde{\Lambda_2}, u\right)$

is a $\mathscr{H}$-graph. Hence the concatenation of the $\mathscr{H}$-path $E_1, ..., E_{r-1}$, the $\mathscr{H}$-path E'_0, E'_1 and of the $\mathscr{H}$-path $F_2, ..., F_s$ delivers a $\mathscr{H}$-path that connects a vertex labeled (Λ_0, v) to a vertex labeled (Λ_2, v).

In any case we proved that Λ_0 is $\mathscr{H}$-equivalent to Λ_2 w.r.t. v. □

Lemma 4.4. *Let Λ_0 be a subgroup of $\mathbb{Z}^d$ and $v \in \mathcal{V}(\mathscr{H})$. There exists a finite $\mathscr{H}$-graph*

- *that is not simply connected;*
- *that contains a vertex labeled (Λ_0, v);*
- *that contains at least two non-saturated vertices.*

Proof. We distinguish two cases:

Case 1: Assume that there exists a loop $e_0 \in \mathcal{E}(\mathscr{H})$ based at v. Let us consider a $\mathscr{H}$-path E_1, E_2 labeled e_0, e_0 such that

- the vertex $\mathbf{t}(E_1) = \mathbf{s}(E_2)$ is labeled (Λ_0, v);
- the vertex $\mathbf{s}(E_1)$ is labeled (Λ_1, v), where $\Lambda_1 = \mathrm{M}_{e,\mathbf{s}} \left(\mathrm{M}_{e,\mathbf{t}}^{-1} \Lambda_0 \cap \mathbb{Z}^d\right)$ (so that Λ_1, Λ_0 and e satisfy the Transfer Equation (3);
- the vertex $\mathbf{t}(E_2)$ is labeled (Λ_2, v), where $\Lambda_2 = \mathrm{M}_{e,\mathbf{t}} \left(\mathrm{M}_{e,\mathbf{s}}^{-1} \Lambda_0 \cap \mathbb{Z}^d\right)$ (so that Λ_0, Λ_2 and e satisfy the Transfer Equation (3).

We apply Lemma 4.2 to get a $\mathscr{H}$-path labeled $e_0, ..., e_0$ with source $\mathbf{t}(E_2)$ and target $\mathbf{s}(E_1)$.

Subcase 1.a: If there exists $f_0 \in \mathcal{E}(\mathscr{H}) \setminus \{e_0, \overline{e_0}\}$, then the vertices $\mathbf{t}(E_2)$ and $\mathbf{s}(E_1)$ are neither saturated relatively to f_0 nor to $\overline{f_0}$, which leads to the conclusion in this particular case.

Subcase 1.b: Otherwize, as G is non-amenable, one has $|\det(\mathrm{M}_{e,\mathbf{s}})| \geq 2$ and $|\det(\mathrm{M}_{e,\mathbf{t}})| \geq 2$ by Proposition 3.18. In particular:

- as $\mathbf{s}(E_1)$ is labeled by a subgroup Λ_1 of $\mathrm{M}_{e,\mathbf{s}}\mathbb{Z}^d$, one has
$$\begin{aligned} \left|\mathbb{Z}^d / \left\langle \Lambda_1, \mathrm{M}_{e,\mathbf{s}}\mathbb{Z}^d \right\rangle\right| &= |\det(\mathrm{M}_{e,\mathbf{s}})| \\ &\geq 2, \end{aligned}$$
so, as E_1 has a single outgoing edge labeled e, the vertex E_1 is non-saturated relatively to e;
- likewise, the vertex $\mathbf{t}(E_2)$ is non-saturated relatively to $\overline{e}$.

Case 2: Otherwise, we fix an edge $e_0 \in \mathcal{E}(\mathscr{H})$ with source v such that $w := \mathbf{t}(e_0) \neq v$. In particular, the graph of groups $\mathscr{H}$ being reduced, denoting by $(\mathrm{A},\mathrm{B}) := (\mathrm{M}_{e_0,\mathbf{s}}, \mathrm{M}_{e_0,\mathbf{t}})$, one has $|\det(A)| \geq 2$ and $|\det(B)| \geq 2$. Let us define a $\mathscr{H}$-path $E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8$ as follows:

- E_1, E_3, E_5, E_7 are labeled e_0 and E_2, E_4, E_6, E_8 are labeled $\overline{e_0}$;
- $\mathbf{s}(E_1)$ is labeled (Λ_0, v);
- the vertices $\mathbf{t}(E_1)$, $\mathbf{t}(E_3)$, $\mathbf{t}(E_5)$ and $\mathbf{t}(E_7)$ are all labeled $(\mathrm{B}(\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d), v)$;
- the vertices $\mathbf{t}(E_2)$, $\mathbf{t}(E_4)$, $\mathbf{t}(E_6)$ and $\mathbf{t}(E_8)$ are labeled $(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d, v)$.

Notice that at least four vertices are non-saturated relatively to some edge $\widehat{e} \in \mathcal{E}(\mathscr{H})$. Indeed, by Proposition 3.18, as G is non-amenable and $\mathscr{H}$ is reduced:

- either $|\det(\mathrm{A})| > 3$. In this case, the vertices $\mathbf{t}(E_2)$, $\mathbf{t}(E_4)$, $\mathbf{t}(E_6)$ and $\mathbf{t}(E_8)$ are non-saturated relatively to e_0;
- or $|\det(\mathrm{B})| \geq 3$. In this case, $\mathbf{t}(E_1)$, $\mathbf{t}(E_3)$, $\mathbf{t}(E_5)$ and $\mathbf{t}(E_7)$ are non-saturated relatively to $\overline{e_0}$;
- or there exists an edge $f_0 \neq e_0$ with source v. In this case, the vertices $\mathbf{t}(E_2)$, $\mathbf{t}(E_4)$, $\mathbf{t}(E_6)$ and $\mathbf{t}(E_8)$ are non-saturated relatively to f_0;
- or there exists an edge $f_0 \neq \overline{e_0}$ with target w. In this case, $\mathbf{t}(E_1)$, $\mathbf{t}(E_3)$, $\mathbf{t}(E_5)$ and $\mathbf{t}(E_7)$ are non-saturated relatively to f_0.

In any case, denoting by V_1, V_2, V_3, V_4 four non-saturated vertices relatively to some edge $\widehat{e}$, Lemma 4.2 delivers a $\mathscr{H}$-graph $\mathcal{C}$ which consists of

- the $\mathscr{H}$-path $E_1, ..., E_8$ as a subgraph;
- another reduced $\mathscr{H}$-path $F_1, ..., F_r$ of type $\widehat{e}, ..., \overline{\widehat{e}}$ with source V_1 and target V_2.

The vertices V_3 and V_4 are non-saturated relatively to $\widehat{e}$ in $\mathcal{C}$. Hence $\mathcal{C}$ is suitable. □

5 Perfect kernel of non-amenable GBS groups

The goal of this section is to give an explicit description of the perfect kernel in the case where the GBS group G is non-amenable, *i.e.* is defined neither by a single loop with at least one invertible matrix nor by a segment with two matrices having determinant ± 2.

We start with the following lemma, that gives an inclusion in a more general setting:

Lemma 5.1. *Let G be the fundamental group of a finite graph of groups $\mathscr{H}$ such that, for every $v \in \mathcal{V}(\mathscr{H})$, the group G_v is noetherian (i.e. every subgroup of G_v is finitely generated (e.g. the G_v's are finitely generated abelian groups)). Let us denote by $\mathcal{T}$ the Bass-Serre tree of $\mathscr{H}$. Then*

$$\mathcal{K}(G) \subseteq \{H \leq G \mid H\backslash\mathcal{T} \text{ is infinite}\}.$$

Proof. Let H be a subgroup of G whose graph of groups $H\backslash\mathcal{T}$ is finite. Let us show that under the assumptions of the lemma, the set

$$\Omega = \{H' \leq G, H \leq H'\}$$

is a countable neighborhood of H in $\mathrm{Sub}(G)$.

First notice that every element of Ω has a finite graph of groups. In particular, any element of Ω is finitely generated: denoting by $\mathscr{K}$ its graph of groups, Bass-Serre theory tells us that it is generated by a finite number of subgroups of G_v's (one per each vertex of $\mathscr{K}$, each of these being finitely generated by noetherianity of the G_v's), and by one element per edge of $\mathscr{K}$. In particular:

- as H belongs to Ω, it is finitely generated so Ω is an open neighborhood of H;
- Ω is included in the subset of finitely generated subgroups, hence is countable.

This proves that H has a countable neighborhood, thus $H \notin \mathcal{K}(G)$. □

Now we prove Theorem 1.1:

Theorem 5.2. *Let G be a non-amenable GBS group defined by a reduced graph of groups $\mathscr{H}$ and let $\mathcal{T}$ be the associated Bass-Serre tree. Then*

$$\mathcal{K}(G) = \{H \leq G \mid H\backslash\mathcal{T} \text{ is infinite}\}.$$

Proof. By Lemma 5.1 and Remark 3.7, it suffices to show that any subgroup H of G whose $\mathscr{H}$-graph $\mathcal{G}$ is infinite belongs to $\mathcal{K}(G)$. Let us fix $v \in \mathcal{V}(\mathscr{H})$. Let us denote by α the associated $\mathscr{H}$-preaction. Let β be a subpreaction of α whose $\mathscr{H}$-graph is a finite subgraph K of $\mathcal{G}$. By assumption, K has a vertex V_0 labeled (Λ_0, v_0) for some $\Lambda_0 \leq \mathbb{Z}^d$ and some $v_0 \in \mathcal{V}(\mathscr{H})$, which is not saturated relatively to some edge e_0 with source v_0. By Lemma 3.14, there exists a $\mathscr{H}$-graph $\mathcal{G}_0$ that contains K and such that the quotient $\mathcal{G}_0/K$ is an infinite forest. Hence, by Lemma 3.11, there exists a $\mathscr{H}$-preaction α_0 that extends β and whose $\mathcal{G}$-graph is $\mathcal{G}_0$.

Let us first assume that $\mathscr{H}$ does not consist of a single vertex and a collection of loops labeled by matrices in $GL_d(\mathbb{Z})$. By Lemma 4.4, there exists a non-simply connected finite $\mathscr{H}$-graph $\mathcal{C}$ that contains

- a vertex labeled (Λ_0, v);
- two vertices V and W labeled (v_1, Λ_1) and (v_2, Λ_2) that are non-saturated relatively to some edges denoted by e, f, respectively.

By Lemma 4.4, $\mathcal{C}$ is the $\mathscr{H}$-graph of a $\mathscr{H}$-preaction γ. Lemma 4.2 delivers a $\mathscr{H}$-path $\mathcal{P}$ of type $e_0, ..., \overline{e}$ that connects V_0 to V. The vertex W is non-saturated relatively to f in the $\mathscr{H}$-graph $K' := K \cup \mathcal{P} \cup \mathcal{C}$. By Lemma 3.14, there exists a $\mathscr{H}$-graph $\mathcal{G}_1$ that contains K' and such that the quotient $\mathcal{G}_1/K'$ is a forest. Hence, by Lemma 3.11, there exists a $\mathscr{H}$-preaction α_1 that extends both β and γ and whose $\mathscr{H}$-graph is $\mathcal{G}_1$. As $\mathcal{G}_0$ and $\mathcal{G}_1$ are non-isomorphic (because they don't share the same homotopy type), the (saturated) $\mathscr{H}$-preactions α_0 and α_1 both extend β and the associated subgroups of G are different.

Otherwise, our group G is of the form $\mathbb{Z}^d \rtimes \mathbb{F}_r$ (where $r \geq 2$ denotes the number of loops) and each generator of $\mathbb{F}_r$ acts on $\mathbb{Z}^d$ by multiplication by an invertible integer matrix. Denoting by $\pi : G = \mathbb{Z}^d \rtimes \mathbb{F}_r \to \mathbb{F}_r$ the canonical surjection, every subgroup of G is fully determined by its intersection $\Lambda_0 = \Lambda \cap \mathbb{Z}^d$ with $\mathbb{Z}^d$, its image $\pi(\Lambda) \leq \mathbb{F}_r$ under π satisfying

$$x \cdot \Lambda_0 = \Lambda_0 \ \forall x \in \pi(\Lambda)$$

and, given a basis $(a_i)_{i \in I}$ (the set I being finite or countable) of $\pi(\Lambda)$, elements $(u_i, a_i) \in \Lambda$ for every $i \in I$. Notice that in this case, the Bass-Serre tree of G is the Cayley graph of $\mathbb{F}_r$ with respect to the standard generating set, and that for any subgroup H of G, the quotient graph $H \backslash \mathcal{T}$ is infinite iff $\pi(H) \in \mathrm{Sub}_{[\infty]}(\mathbb{F}_r)$. One distinguishes two cases:

1. Let us first assume that $\mathrm{rk}(\Lambda_0) = d$. Let us denote by $(\mathrm{A}, \mathrm{B}) = (\mathrm{M}_{e_0,\mathbf{s}}, \mathrm{M}_{e_0,\mathbf{t}})$, let us consider an edge $f_0 \neq e_0$ and let us write $(\mathrm{C}, \mathrm{D}) = (\mathrm{M}_{f_0,\mathbf{s}}, \mathrm{M}_{f_0,\mathbf{t}})$. As the subgroup of $SL_2(\mathbb{Z})$ generated by DC^{-1} acts on the (finite) set of lattices of determinant $\pm \det(\Lambda_1)$, there exists an integer $k \in \mathbb{N}^*$ satisfying

   $$(\mathrm{DC}^{-1})^k \Lambda_0 = \Lambda_0$$

   In particular, there exists a $\mathscr{H}$-cycle $\mathcal{C} = E_1, ..., E_k$ all of whose edges are of type f_0 such that $\mathbf{s}(E_i)$ is labeled $(\mathbf{s}(e_0), (\mathrm{DC}^{-1})^i \Lambda_0)$ for every $i \in [\![1, k]\!]$. As no vertex of $\mathcal{C}$ is saturated relatively to e_0, Lemma 4.2 delivers a $\mathscr{H}$-path $\mathcal{P}$ of type $e_0, ..., \overline{e_0}$ that connects V_0 to a vertex V of $\mathcal{C}$. Any other vertex of $\mathcal{C}$ is not saturated relatively to e_0 in $K' := K \cup \mathcal{P} \cup \mathcal{C}$. Hence we can conclude as in the previous case.

2. If $\mathrm{rk}(\Lambda_0) < d$, let us write Λ_0 in its Smith normal form:

   $$\Lambda_0 = P\mathrm{diag}(d_1, ..., d_r, 0..., .0)\mathbb{Z}^d$$

(with $P \in GL_d(\mathbb{Z})$, $r < d$ and $d_i | d_{i+1}$ for every $i \in [\![1, r-1]\!]$). For every $N \in \mathbb{N}$, define

$$\Lambda_0^{(N)} = P\mathrm{diag}\left(d_1, ..., d_r, N\prod_{i=1}^r d_i, ..., N\prod_{i=1}^r d_i\right)\mathbb{Z}^d$$

so that

- every matrix stabilizing the subgroup Λ_0 also stabilizes $\Lambda_0^{(N)}$ (in particular, $\Lambda_0^{(N)}$ is $\pi(\Lambda)$-stable);
- the sequence $\left(\Lambda_0^{(N)}\right)_{N\in\mathbb{N}}$ tends to Λ_0 in $\mathrm{Sub}(\mathbb{Z}^d)$ as N tends to $+\infty$.

Moreover, as $(a_i)_{i\in I}$ is a free basis of $\langle a_i, i \in I\rangle$, one has

$$\left(\mathbb{Z}^d \times 1\right) \cap \langle (u_i, a_i), i \in I\rangle = \{1\}.$$

This implies that the sequence of subgroups

$$\begin{aligned}\widetilde{\Lambda}_N &= \langle \Lambda_0^{(N)}, (u_i, a_i)_{i\in I}\rangle \\ &= \Lambda_0^{(N)} \rtimes \langle (u_i, a_i)_{i\in I}\rangle\end{aligned}$$

converges to Λ (non-trivially, because $\widetilde{\Lambda}_N \cap \mathbb{Z}^d = \Lambda_0^{(N)}$). As $\pi(\widetilde{\Lambda}_N) = \pi(\Lambda)$, the $\mathscr{H}$-graphs of $\widetilde{\Lambda}_N$ and Λ have isomorphic skeletons (in particular, the subgroups $\widetilde{\Lambda}_N$ have infinite $\mathscr{H}$-graphs).

□

Remark 5.3. Notice that we didn't make use of the cocompactness of the action $G \curvearrowright \mathcal{T}$. Thus, Theorem 5.2 extends to a larger class of groups that incluse GBS groups of rank d, *i.e.* the class of non-amenable groups acting (non necessarily cocompactly) on an oriented tree with vertex and edge stabilizers isomorphic to $\mathbb{Z}^d$. In particular, in the case of a non-cocompact action on the Bass-Serre tree, the perfect kernel consists of the whole set of subgroups $\mathrm{Sub}(G)$.

The following corollary gives a class of GBS groups that satisfy the equality $\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G)$. Recall that in rank 1, the authors of [CGMS25] and [Bon24] proved that this equality was true for non-unimodular GBS groups only. Let v be a vertex of $\mathscr{H}$ and recall that the modular homomorphism $\Delta_G^{(v)}$ based at v is defined by the following data:

- $\Delta_G^{(v)}$ is trivial on the vertex groups;

- for every edge generator t_e, denoting by $e_1, ..., e_r$ the unique reduced edge path in $\mathscr{T}$ with source v and target $\mathbf{s}(e)$, and by $e_{r+1}, ...e_s$ the unique reduced edge path in $\mathscr{T}$ with source $\mathbf{t}(e)$ and target v, one has

$$\Delta_G^{(v)}(t_e) = \mathrm{M}_{e_s,\mathbf{t}}\mathrm{M}_{e_s,\mathbf{s}}^{-1} \dots \mathrm{M}_{e_1,\mathbf{t}}\mathrm{M}_{e_1,\mathbf{s}}^{-1}.$$

Corollary 5.4. *Let G be a non-amenable and non-unimodular GBS group satisfying the following property: for every non-trivial infinite index subgroup $\Lambda_0 \in \mathrm{Sub}_{[\infty]}(\mathbb{Z}^d)$, the subgroup*

$$\{g \in G \mid \Delta_G^{(v)}(g) \cdot \Lambda_0 \ \text{ is commensurable to } \ \Lambda_0\}$$

has infinite index in G. Then

$$\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G).$$

Remark 5.5. If Λ_0 is trivial or has finite index in $\mathbb{Z}^d$, then $\Delta_G^{(v)}(g) \cdot \Lambda_0$ is commensurable to Λ_0 for every $g \in G$.

Before proving Corollary 5.4, we give an explicit example of a GBS group G of rank 2 whose perfect kernel consists of the set of infinite index subgroups of G. Let us define $\mathrm{A} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and let us define $G = \pi_1(\mathscr{H})$ as the fundamental group of the graph of groups defined in Figure 7. In other words, G is defined by the following presentation:

$$G \simeq \left\langle x, y, t \mid xy = yx, t^{-1}x^2y^2t = x, t^{-1}x^2y^4t = y^2 \right\rangle.$$

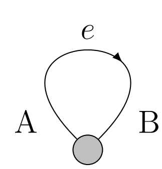

Figure 7: The graph of groups $\mathscr{H}$.

The image of the modular homomorphism Δ_G is the subgroup of $GL_2(\mathbb{Q})$ generated by $\mathrm{B}^{-1}\mathrm{A} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$. Hence $\mathrm{Im}(\det \circ \Delta_G) = 2^{\mathbb{Z}}$, thus G is not unimodular. Notice that $\mathrm{Spec}_{\mathbb{R}}(\mathrm{B}^{-1}\mathrm{A}) = \{2 + \sqrt{2}, 2 - \sqrt{2}\}$, so $\mathrm{Spec}_{\mathbb{R}}((\mathrm{B}^{-1}\mathrm{A})^n) = \left\{\left(2 + \sqrt{2}\right)^n, \left(2 - \sqrt{2}\right)^n\right\}$ for every $n \in \mathbb{Z}$. Thus, for any $n \in \mathbb{Z} \setminus \{0\}$, one has $\mathrm{Spec}_{\mathbb{Q}}((\mathrm{B}^{-1}\mathrm{A})^n) = \varnothing$. In particular, for every $u \in \mathbb{Q}^2 \setminus \{0\}$, the set

$$\{(\mathrm{B}^{-1}\mathrm{A})^n\mathbb{Q}u, n \in \mathbb{Z}\}$$

is infinite, up to commensurability. Hence, by Corollary 5.4, one has $\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G)$.

Now let us prove Corollary 5.4. To begin with, we prove the following lemma:

Lemma 5.6. *Let H be a subgroup of G and let us denote by $\mathcal{G}$ its $\mathscr{H}$-graph. Let $v \in \mathcal{V}(\mathscr{H})$ and let (v, Λ_0) be the label of a vertex V_0 of $\mathcal{G}$. Then, up to commensurability:*

$$\begin{aligned}\{\Delta_G^{(v)}(g)\cdot\Lambda_0 \mid g \in G\} &= \{\Lambda_1 \leq \mathbb{Z}^d \mid \exists a \text{ vertex } V \in \mathcal{V}(\mathcal{G}) \text{ labeled } (\Lambda_1, v)\}\\ &= \{\Lambda_1 \leq \mathbb{Z}^d \mid \Lambda_1 \text{ is } \mathscr{H}\text{-equivalent to } \Lambda_0 \text{ w.r.t. } v.\}\end{aligned}$$

Proof. Let us prove that

$$\{\Delta_G^{(v)}(g)\cdot\Lambda_0 \mid g \in G\} \subseteq \{\Lambda_1 \leq \mathbb{Z}^d \mid \exists \text{a vertex } V \in \mathcal{V}(\mathcal{G}) \text{ labeled } (\Lambda_1, v)\}.$$

Let $g \in G$. Let us consider a cycle $e_1, ..., e_r$ based at v in $\mathscr{H}$ and elements $g_i \in G_{\mathbf{s}(e_i)}$ for every i and $g_{r+1} \in G_s$ such that

$$g = g_1 \cdot s_{e_1} \cdot g_2 \ldots \cdot s_{e_r} \cdot g_{r+1}$$

(where $s_{e_i} = t_{e_i}$ if $e_i \notin \mathscr{T}$ and 1 otherwise). Denoting by $(\mathrm{A}_i, \mathrm{B}_i)$ the label of e_i for every $i \in [\![1, r]\!]$, one has:

$$\Delta_G^{(v)}(g) = \mathrm{B}_r\mathrm{A}_r^{-1}\ldots\mathrm{B}_1\mathrm{A}_1^{-1}. \tag{5.7}$$

Let us consider an edge path $E_1, ..., E_r$ based at V_0 and labeled $e_1, ...e_r$ in $\mathcal{G}$. For every $i \in [\![1, r]\!]$, let $(\Lambda_i, \mathbf{t}(e_i))$ be the label of $\mathbf{t}(E_i)$. By the Transfer Equation 3, we get

$$\mathrm{A}_i^{-1}\Lambda_{i-1} \cap \mathbb{Z}^d = \mathrm{B}_i^{-1}\Lambda_i \cap \mathbb{Z}^d$$

which implies that

$$\mathrm{A}_i^{-1}\Lambda_{i-1} \otimes \mathbb{Q}^d = \mathrm{B}_i^{-1}\Lambda_i \otimes \mathbb{Q}^d.$$

Thus

$$\Lambda_r \otimes \mathbb{Q}^d = \mathrm{B}_r\mathrm{A}_r^{-1}\ldots\mathrm{B}_1\mathrm{A}_1^{-1}\Lambda_0 \otimes \mathbb{Q}^d \tag{5.8}$$

which implies, with Equation 5.7, that Λ_r is commensurable to $\Delta_G^{(v)}(g)\cdot\Lambda_0$.

If there exists a vertex $V \in \mathcal{V}(\mathcal{G})$ labeled (Λ_1, v), then Λ_1 is $\mathscr{H}$-equivalent to Λ_0 w.r.t. v by connectedness of $\mathcal{G}$. This proves that

$$\begin{aligned}&\{\Lambda_1 \leq \mathbb{Z}^d \mid \exists \text{a vertex } V \in \mathcal{V}(\mathcal{G}_\Lambda) \text{ labeled } (\Lambda_1, v)\}\\ &\subseteq \{\Lambda_1 \leq \mathbb{Z}^d \mid \Lambda_1 \text{ is } \mathscr{H}\text{-equivalent to } \Lambda_0 \text{ w.r.t. } v\}.\end{aligned}$$

Finally, let us prove that

$$\{\Lambda_1 \leq \mathbb{Z}^d \mid \Lambda_1 \text{ is } \mathscr{H}\text{-equivalent to } \Lambda_0 \text{ w.r.t. } v\} \subseteq \{\Delta_G^{(v)}(g)\cdot\Lambda_0, g \in G\}$$

up to commensurability. Let Λ_1 be a subgroup which is $\mathscr{H}$-equivalent to Λ_0 w.r.t. v. Let us consider a $\mathscr{H}$-path $E_1,...,E_r$ labeled $e_1,...,e_r$ whose source is labeled (Λ_0, v) and whose target is labeled (Λ_1, v). For $i \in [\![1,r]\!]$, let us denote by $(\Gamma_i, \mathbf{s}(e_i))$ the label of $\mathbf{s}(E_i)$ (hence $\Gamma_1 = \Lambda_0$). By the Transfer Equation 3, denoting by $(\mathrm{A}_i, \mathrm{B}_i)$ the label of e_i in $\mathscr{H}$ we get, as previously

$$\Lambda_1 \otimes \mathbb{Q}^d = \mathrm{B}_r\mathrm{A}_r^{-1}\dots\mathrm{B}_1\mathrm{A}_1^{-1}\Lambda_0 \otimes \mathbb{Q}^d.$$

Thus, denoting by $g = s_{e_1}...s_{e_r}$ (where $s_{e_i} = t_{e_i}$ if $e_i \notin \mathscr{T}$ and 1 otherwise), one has:

$$\Lambda_1 \otimes \mathbb{Q}^d = \Delta_G^{(v)}(\gamma) \cdot \Lambda_0 \otimes \mathbb{Q}^d,$$

which proves that Λ_1 is commensurable to $\Delta_G^{(v)}(g) \cdot \Lambda_0$. □

Proof of Corollary 5.4. Let H be a subgroup of G whose graph of groups is finite. Let us show that H has finite index under the assumptions of the corollary. Let $\mathcal{G}$ be the $\mathscr{H}$-graph of H. Let us denote by (Λ_0, v) the label of a vertex of $\mathcal{G}$. By Lemma 5.6, the orbit of the commensurability class of Λ_0 under the action of $\mathrm{Im}\left(\Delta_G^{(v)}\right)$ is finite. Hence its stabilizer

$$\{g \in G \mid \Delta_G^{(v)}(g) \cdot \Lambda_0 \text{ is commensurable to } \Lambda_0\}$$

has finite index in G. This implies that

- either Λ_0 has finite index in $\mathbb{Z}^d$;
- or $\Lambda_0 = 0$.

Let us assume by contradiction that $\Lambda_0 = \{0\}$. By commensurability of the vertex stabilizers, this is equivalent to

$$H \cap G_w = \{1\} \text{ for every vertex } w \in \mathcal{V}(\mathscr{H})$$

(or equivalently, all the labels of the vertices of $\mathcal{G}$ are trivial).

As the image of the morphism $\left|\det \circ \Delta_G^{(v)}\right|$ is non-trivial, there exists a (reduced) cycle of edges $e_1,...,e_n$ in $\mathscr{H}$ based at v and satisfying

$$\left|\frac{\prod_{i=1}^n \det(\mathrm{A}_i)}{\prod_{i=1}^n \det(\mathrm{B}_i)}\right| \neq 1$$

Let n_i be the number of vertices of $\mathcal{G}$ labeled $\mathbf{s}(e_i)$. Every vertex labeled $\mathbf{s}(e_i)$ (*resp.* $\mathbf{t}(e_i)$) has $|\det(\mathrm{A}_i)|$ (*resp.* $|\det(\mathrm{B}_i)|$) outgoing (*resp.* incoming) edges labeled e_i. Hence the number of edges labeled e_i in $\mathcal{G}$ is

$$n_i|\det(\mathrm{A}_i)| = n_{i+1}|\det(\mathrm{B}_i)|.$$

Combining these equalities for $i = 1, ..., n$, we get

$$\prod_{i=1}^{n} |\det(\mathrm{A}_i)| = \prod_{i=1}^{n} |\det(\mathrm{B}_i)|$$

hence a contradiction. Thus, Λ_0 has finite index in $G_{\mathbf{s}(e_1)}$. By commensurability of the vertex stabilisers, this implies that $H \cap G_w$ has finite index in G_w for every $w \in \mathcal{V}(\mathscr{H})$ which implies that $H\backslash G$ is finite (because $\mathcal{G}$ is also finite).

Using Theorem 5.2, we finally deduce the equality

$$\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G).$$

□

Remark 5.9. If G is a non-unimodular GBS group such that every element of the image of the modular homomorphism $\Delta_G^{(v)}$ is irreducible (as a $\mathbb{Q}$-linear endomorphism of $\mathbb{Q}^d$), then the assumptions of Corollary 5.4 are satisfied. Let us explain why. As G is non-unimodular, there exists $g \in G$ such that $\left|\det\left(\Delta_G^{(v)}(g)\right)\right| \neq 1$. In particular, $\Delta_G^{(v)}(g)$ has infinite order, hence $\mathrm{Im}\left(\Delta_G^{(v)}\right)$ is infinite. Now assume by contradiction that there exists a non-trivial infinite index subgroup Λ_0 of $\mathbb{Z}^d$ such that the subgroup

$$\left\{g \in G \mid \Delta_G^{(v)}(g) \cdot \Lambda_0 \ \text{ is commensurable to } \ \Lambda_0\right\}$$

has finite index in G. Equivalently, the orbit of the commensurability class of Λ_0 under the action of $\mathrm{Im}\left(\Delta_G^{(v)}\right)$ is finite, so by the piegonhole principle, there exists a non-trivial element $\Delta_G^{(v)}(g) \in \mathrm{Im}\left(\Delta_G^{(v)}\right)$ such that $\Delta_G^{(v)}(g) \cdot (\Lambda_0 \otimes \mathbb{Q}) = \Lambda_0 \otimes \mathbb{Q}$. In particular, $\Lambda_0 \otimes \mathbb{Q}^d$ is a non-trivial subspace of the $\mathbb{Q}$-vector space $\mathbb{Q}^d$ which is stable under $\Delta_G^{(v)}(g)$, which contradicts the assumption made on G.

6 A dynamical partition of the perfect kernel

The goal of this section is to extend the decomposition of the perfect kernel obtained in [CGMS25] and in [Bon24] for non-amenable GBS groups of rank 1.

6.1 Case where G is not a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$

In this section, we assume that G is not a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$, *i.e.* $\mathscr{H}$ does not consist of a single vertex with a collection of loops all of whose labels are in $GL_d(\mathbb{Z})$. This allow us to make use of the equivalence relation on $\mathrm{Sub}(\mathbb{Z}^d)$ defined in 4.

Let us fix a vertex $v \in \mathcal{V}(\mathscr{H})$. We identify G_v with $\mathbb{Z}^d$. Let us denote by $\simeq$ the $\mathscr{H}$-equivalence relation with respect to v and by $\pi_v : \mathrm{Sub}(\mathbb{Z}^d) \to \mathrm{Sub}(\mathbb{Z}^d)/\simeq$ the canonical projection. Notice that the rank is constant on each fiber of π_v. We have the following statement:

Proposition 6.1. *The set* $\mathrm{Sub}(\mathbb{Z}^d)/\simeq$ *is infinite countable.*

Proof. As $\mathrm{Sub}(\mathbb{Z}^d)$ is countable, the set $\mathrm{Sub}(\mathbb{Z}^d)/\simeq$ is also countable. Let us show that it is infinite. Let us define the finite subset $\mathcal{P}_{\mathscr{H}}$ of prime numbers

$$\mathcal{P}_{\mathscr{H}} = \{p \in \mathcal{P} \mid \text{there exists an edge } e \in \mathcal{E}(\mathscr{H}), p \mid \det(\mathrm{M}_{e,\mathbf{t}})\}.$$

Let us define

$$\delta : \begin{array}{ccc} \mathcal{L}(\mathbb{Z}^d) & \to & \mathbb{Z} \\ \Lambda & \mapsto & \prod_{p \notin \mathcal{P}_{\mathscr{H}}} p^{|\det(\Lambda)|_p} \end{array}.$$

The image of δ is exactly the set of integers which are divisible by no element of $\mathcal{P}_{\mathscr{H}}$, hence is infinite. Let us show that δ is constant on the fibers of π_v, *i.e.* that we have a factorization

$$\begin{tikzcd} \mathcal{L}(\mathbb{Z}^d) \arrow[r, "\delta"] \arrow[dr, "\pi_v"'] & \delta(\mathbb{Z}) \\ & \mathcal{L}(\mathbb{Z}^d)/\simeq \arrow[u, "\bar{\delta}"'] \end{tikzcd}$$

which will imply that $\mathcal{L}(\mathbb{Z}^d)/\simeq$ is infinite (hence $\mathrm{Sub}(\mathbb{Z}^d)/\simeq$ is infinite).

By a straightforward induction on the length of a $\mathscr{H}$-path, it suffices to prove that for any $\mathscr{H}$-edge labeled $e \in \mathcal{E}(\mathscr{H})$ with source labeled $(\Lambda_0, \mathbf{s}(e))$ and target labeled $(\Lambda_1, \mathbf{t}(e))$, one has $\delta(\Lambda_0) = \delta(\Lambda_1)$. Let E be such an edge. By the Transfer Equation 3, denoting by $(\mathrm{A}, \mathrm{B}) = (\mathrm{M}_{e,\mathbf{s}}, M_{e,\mathbf{t}})$, one has

$$\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d = \mathrm{B}^{-1}\Lambda_1 \cap \mathbb{Z}^d.$$

Let $p \in \mathcal{P} \setminus \mathcal{P}_{\mathscr{H}}$. We have

$$\frac{\det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d)}{\det(A)} = \frac{\det(\Lambda_1 \cap \mathrm{B}\mathbb{Z}^d)}{\det(B)},$$

hence, as $p \nmid \det(\mathrm{A})$ and $p \nmid \det(\mathrm{B})$:

$$|\det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d)|_p = |\det(\Lambda_1 \cap \mathrm{B}\mathbb{Z}^d)|_p. \tag{6.2}$$

From $\det(A)\Lambda_0 \leq \Lambda_0 \cap \mathrm{A}\mathbb{Z}^d \leq \Lambda_0$, we get

$$\det(\Lambda_0) \mid \det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d) \mid \det(\Lambda_0)(\det(\mathrm{A}))^d.$$

Thus, as $p \nmid \det(\mathrm{A})$:

$$|\det(\Lambda_0)|_p = |\det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d)|_p.$$

Likewise

$$|\det(\Lambda_1)|_p = |\det(\Lambda_1 \cap \mathrm{B}\mathbb{Z}^d)|_p,$$

so by Equation 6.2 we finally get

$$|\det(\Lambda_0)|_p = |\det(\Lambda_1)|_p.$$

As this is true for any $p \notin \mathcal{P}_{\mathscr{H}}$, this implies that

$$\delta(\Lambda_0) = \delta(\Lambda_1)$$

as required. $\square$

Let us define the **$\mathscr{H}$-phenotype with respect to** v as the following function:

$$\mathrm{Ph}_{\mathscr{H},v} : \begin{array}{ccc} \mathrm{Sub}(G) & \to & \mathrm{Sub}(\mathbb{Z}^d)/\simeq \\ H & \mapsto & \pi_v(H \cap G_v) \end{array}.$$

Proposition 6.3. *The $\mathscr{H}$-phenotype $\mathrm{Ph}_{\mathscr{H},v}$ is surjective and invariant under conjugation by any element of G.*

Proof. The surjectivity of $\mathrm{Ph}_{\mathscr{H},v}$ results from the surjectivity of π_v and of the function $\begin{array}{ccc} \mathrm{Sub}(G) & \to & \mathrm{Sub}(G_v) \\ H & \mapsto & H \cap G_v \end{array}$.

Let H be a subgroup of G and $g \in G$. Let $\Lambda_0 \coloneqq H \cap G_v \leq \mathbb{Z}^d$ and $\Lambda_1 \coloneqq gHg^{-1} \cap G_v \leq \mathbb{Z}^d$. Let $\mathcal{G}$ be the $\mathscr{H}$-graph of H. By definition, there exist two vertices labeled (Λ_0, v) and (Λ_1, v) in $\mathcal{G}$. Hence, by connectedness of $\mathcal{G}$, one has $\pi_v(\Lambda_0) = \pi_v(\Lambda_1)$. $\square$

As the group G_v acts non-cocompactly on the Bass-Serre tree $\mathcal{T}$, any subgroup of G_v lies in the perfect kernel $\mathcal{K}(G)$ by Theorem 5.2. In particular, the restriction $\begin{array}{ccc} \mathcal{K}(G) & \to & \mathrm{Sub}(G_v)/\simeq \\ H & \mapsto & \pi_v(H \cap G_v) \end{array}$ remains surjective and invariant under conjugation.

This function leads to a dynamical partition of the perfect kernel

$$\mathcal{K}(G) = \bigsqcup_{\Lambda \leq \mathbb{Z}^d} \mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda)). \tag{6.4}$$

By the previous remark and Proposition 6.1, this partition is infinite countable. Now we are able to prove Theorem 1.2 which gives a description of the dynamics induced on each piece of the aforementioned decomposition:

Theorem 6.5. *For any $\Lambda_0 \leq \mathbb{Z}^d$, there exists a dense orbit in $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0)) \cap \mathcal{K}(G)$.*

Proof. Let $\Lambda \leq \mathbb{Z}^d$ and let $(H_i)_{i\in\mathbb{N}} \in \left(\mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))\right)^{\mathbb{N}}$ be the sequence of finitely generated subgroups lying in $\mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))$. For $i \in \mathbb{N}$, let α_i be the $\mathscr{H}$-action associated to H_i and let $\mathcal{G}_i$ be the $\mathscr{H}$-graph of α_i. Let β_i be a subpreaction of α_i that corresponds to the same subgroup as α_i and whose $\mathscr{H}$-graph K_i is finite (legit, because H_i is finitely generated). As $H_i \in \mathcal{K}(G)$, Theorem 5.2 implies that there exists a vertex $V_i \in \mathcal{V}(K_i)$ labeled (Λ_i, v_i) which is non-saturated relatively to some edge $e_i \in \mathcal{E}(\mathscr{H})$. Up to extending β_i, one can assume that $v_i = v$. In particular, the subgroups $(\Lambda_i)_{i\in\mathbb{N}}$ of $\mathbb{Z}^d$ are pairwise $\mathscr{H}$-equivalent w.r.t. v so by Lemma 4.2, there exists a $\mathscr{H}$-path $E_{1,i}, ..., E_{r_i,i}$ labeled $e_i, ..., \overline{e_{i+1}}$ that connects V_i to V_{i+1}. Denoting by $\mathcal{G}$ the resulting $\mathscr{H}$-graph, Lemma 3.14 delivers a $\mathscr{H}$-graph $\mathcal{F}$

- that contains $(K_i)_{i\in\mathbb{N}}$ as disjoint subgraphs;
- such that the quotient $\mathcal{F}/\bigsqcup_{i\in\mathbb{N}} K_i$ is a forest.

Hence, by Lemma 3.11, there exists a $\mathscr{H}$-action β that extends β_i for every $i \in \mathbb{N}$ and whose $\mathscr{H}$-graph is $\mathcal{F}$. This proves that the conjugacy class of G which is associated to β is dense in $\mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))$. ☐

Now we study the topology of the pieces of the partition

$$\mathrm{Sub}(G) = \bigsqcup_{\Lambda\leq\mathbb{Z}^d} \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda)).$$

Proposition 6.6. *For any $\Lambda_0 \leq \mathbb{Z}^d$:*

1. *if Λ_0 has finite index in $\mathbb{Z}^d$, then the fiber $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is open;*
2. *otherwise, $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is an F_σ;*
3. *$\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\{\Lambda_0\}))$ is closed iff $\{\Lambda_1 \leq \mathbb{Z}^d \mid \Lambda_1 \simeq \Lambda_0\}$ is finite.*

Remark 6.7. In particular, $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\{0\}))$ is closed.

Remark 6.8. If the image of the modular homomorphism is trivial, then the third item of Lemma 6.6 together with Lemma 5.6 implies that all pieces are closed.

Proof of Proposition 6.6. Notice that we have

$$\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0)) = \bigcup_{\Lambda_1\simeq\Lambda_0} \{H \leq G \mid H \cap G_v = \Lambda_1\}.$$

By Lemma 2.1, this is a F_σ as a countable union of closed subsets of $\mathrm{Sub}(G)$. Hence, we get the second point.

If Λ_0 has finite index in $\mathbb{Z}^d$, this is an open subset of $\mathrm{Sub}(G)$ as a union of open sets by Lemma 2.1. This proves the first point.

If $\{\Lambda_1 \le \mathbb{Z}^d \mid \Lambda_1 \simeq \Lambda_0\}$ is finite, then $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is closed as a finite union of closed subsets by Lemma 2.1. Otherwise, there exists a sequence of subgroups $(\Lambda_n)_{n\in\mathbb{N}} \in (\pi_v^{-1}(\pi_v(\Lambda_0)))^{\mathbb{N}}$ that converges to a subgroup $\Lambda \le \mathbb{Z}^d$ whose rank is strictly less than $\mathrm{rk}(\Lambda_0)$. By a straightforward induction using Corollary 4.3, there exists a $\mathscr{H}$-path $E_1, ..., E_n, ...$ such that: for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $\mathbf{s}\,(E_{k_n})$ is labeled (Λ_n, v). Thus, by Lemma 3.14, there exists a saturated $\mathscr{H}$-graph which is a forest that contains $E_1, ..., E_n, ...$ as a sub-$\mathscr{H}$-graph. Hence by Lemma 3.11, there exists a subgroup $H \le G$ and elements $g_n \in G$ such that $g_n H g_n^{-1} \cap G_v = \Lambda_n$ for every $n \in \mathbb{N}$. Up to extracting, the sequence $g_n H g_n^{-1}$ converges to a subgroup $K \le G$ that satisfies $K \cap G_v = \Lambda$. In particular, as the rank is constant on the fibers of π_v, one has $K \notin \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))$, which proves that $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is not closed. $\square$

Remark 6.9. The definition of the equivalence relation $\simeq$ still makes sense if the graph $\mathscr{H}$ is infinite and the same proof extends to the class of groups that act (non necessarily cocompactly) on an oriented tree with vertex and edge stabilizers isomorphic to $\mathbb{Z}^d$. In the case of a non-cocompact action on the Bass-Serre tree, we thus obtain a dynamical partition of the whole set of subgroups by Remark 5.3.

6.2 Case where $G = \mathbb{Z}^d \rtimes \mathbb{F}_r$

Now we assume that $\mathscr{H}$ consists of a collection of r loops $e_1, ..., e_r$ based at a single vertex v such that, for every $i \in [\![1, r]\!]$, the label $(\mathrm{A}_i, \mathrm{B}_i)$ of e_i satisfies: $\mathrm{A}_i \in GL_d(\mathbb{Z})$ and $\mathrm{B}_i \in GL_d(\mathbb{Z})$. Denoting by

$$P_i \coloneqq \mathrm{A}_i \mathrm{B}_i^{-1},$$

the group G is isomorphic to the semidirect product of $G_v = \mathbb{Z}^d$ with the free group $\mathbb{F}_r = \langle a_1, ..., a_r \rangle$ of rank r, where the generator a_i acts on $\mathbb{Z}^d$ by multiplication by P_i. Let us denote by $\rho : \mathbb{F}_r \to GL_d(\mathbb{Z})$ the morphism that satisfies $\rho(a_i) = P_i$ for every $i \in [\![1, r]\!]$ and by $\Gamma = \rho(\mathbb{F}_r)$. Observe that this $\mathbb{F}_r$-action induces an $\mathbb{F}_r$-action on $\mathrm{Sub}(G_v)$ defined as follows: for any subgroup $\Lambda \le G_v$ and any $\gamma \in \mathbb{F}_r$:

$$\begin{aligned} \gamma \cdot \Lambda &= \rho(\gamma)\Lambda \\ &= (u, \gamma)\Lambda(u, \gamma)^{-1} \ \forall u \in \mathbb{Z}^d \end{aligned}$$

(when identifying G_v with $G_v \times \{1\}$ in $G_v \rtimes \mathbb{F}_r$).

Remark 6.10. If Λ is a finite index subgroup of G_v, then for every $\gamma \in \mathbb{F}_r$, one has $|\det(\rho(\gamma)\Lambda)| = |\det(\Lambda)|$. As there exist only finitely many lattices of $\mathbb{Z}^d$ of a given determinant, the orbit $\mathbb{F}_r \cdot \Lambda = \{\rho(\gamma)\Lambda \mid \gamma \in \mathbb{F}_r\}$ is finite.

Let us denote by $\pi : \mathbb{Z}^d \rtimes \mathbb{F}_r \to \mathbb{F}_r$ the projection.

Our goal is to decompose the perfect kernel of G into countably many pieces on which the action by conjugation contains a dense orbit. Let us recall that (by Theorem 5.2), one has

$$\mathcal{K}(G) = \{H \leq G \mid \pi(H) \in \mathrm{Sub}_{[\infty]}(\mathbb{F}_r)\}.$$

Notice that, as the subgroup G_v is normal, denoting by $Conj(G_v)$ the set of classes for the action of G on $\mathrm{Sub}(G_v)$ by conjugation, the following partition

$$\mathrm{Sub}(G) = \bigsqcup_{\mathscr{C} \in Conj(G_v)} \{H \leq G \mid H \cap G_v \in \mathscr{C}\} \tag{6.11}$$

is G-invariant. Denoting by $\mathscr{C} = \{\rho(\gamma)\Lambda_0, \gamma \in \mathbb{F}_r\}$ for some $\Lambda_0 \leq G_v$, we get that

$$\{H \leq G \mid H \cap G_v \in \mathscr{C}\} = \bigcup_{\gamma \in \mathbb{F}_r} \{H \leq G \mid H \cap G_v = \rho(\gamma)\Lambda_0\}.$$

Notice that $|\det|$ is constant on $\mathscr{C}$ for every $\mathscr{C} \in Conj(G_v)$. In particular, there are infinitely many pieces in the decomposition (6.11).

This partition leads to a G-invariant partition of the perfect kernel into countably many pieces

$$\mathcal{K}(G) = \bigsqcup_{\mathscr{C} \in Conj(G_v)} \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C}\}.$$

Notice that this is exactly the decomposition (6.4) we obtained in the previous case of a GBS group G which is not $\mathbb{Z}^d$-by-free: two subgroups Λ_0, Λ_1 of G_v can arise as the labels of two vertices of some connected $\mathscr{H}$-graph iff there exists some $\gamma \in \mathbb{F}_r$ such that $\rho(\gamma)\Lambda_0 = \Lambda_1$, or equivalently, iff Λ_0 and Λ_1 belong to the same orbit under the G-conjugation. However, the proof of Theorem 6.5 does not extend to our new setting, because the relation $\simeq$ need not be transitive anymore. This difficulty turns out to be a real obstruction: the conjugation action need not be topologically transitive on each piece in our new setting. This comes from the fact that the skeleton of the $\mathscr{H}$-graph of a subgroup of G is related to its intersection with the vertex group G_v as follows:

Lemma 6.12. *Let H be a subgroup of G. Then $\pi(H) \leq \mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v)$.*

Proof. Let $\gamma \in \pi(H)$. There exists $u \in \mathbb{Z}^d$ such that $(u,\gamma) \in H$. For any $(v,1) \in H \cap G_v$, one has

$$(u,\gamma)(v,1)(u,\gamma)^{-1} = (\rho(\gamma)(v),1) \\ \in H$$

which implies that $\rho(\gamma)(v) \in H \cap G_v$. Consequently, $\rho(\gamma)(H \cap G_v) = H \cap G_v$ thus $\gamma \in \mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v)$. Hence, $\pi(H) \leq \mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v)$. $\square$

Let us fix $\mathscr{C} \in Conj(G_v)$. Lemma 6.12 allows us to decompose

$$\begin{aligned}\mathcal{P}_{\mathscr{C}} := &\{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C}\} \\ = &\{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \in \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\} \\ &\bigsqcup \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \notin \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\},\end{aligned}$$

each of these two pieces being invariant under G-conjugation. Notice that the second piece

$$\mathcal{D}_{\mathscr{C}} = \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \notin \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\}$$

of this last decomposition is always countable and open for the induced topology on $\mathcal{P}_{\mathscr{C}}$.

Remark 6.13. More precisely, two cases can occur:

- If $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ is infinitely generated or $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$ (for some, equivalently for all $\Lambda_0 \in \mathscr{C}$), then $\mathcal{D}_{\mathscr{C}}$ is empty;
- Otherwise, it consists of

$$\{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \notin \mathrm{Sub}_{[\infty]}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\}$$

  if $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ is not infinite cyclic, and of $\mathcal{P}_{\mathscr{C}}$ otherwize.

Lemma 6.14. *For every $\mathscr{C} \in Conj(G_v)$, there exists a dense orbit in*

$$\mathcal{P}_{\mathscr{C}} \smallsetminus \mathcal{D}_{\mathscr{C}} = \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \textit{ and } \ \pi(H) \in \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\}.$$

To prove this lemma we will use the formalism of $\mathscr{H}$-graphs. Notice that in this context, the $\mathscr{H}$-graph $\mathcal{G}$ of a subgroup $H \leq G$ will be uniquely determined by $\pi(H)$ and $H \cap G_v$. It is the Schreier graph of the subgroup $\pi(H) \leq \mathbb{F}_r$ (with respect to the generating set $\{a_1, \ldots, a_r\}$) whose labels are defined as follows: if V_0 is a vertex labeled $\Lambda_0 := H \cap G_v$, and V is any other vertex of $\mathcal{G}$, then, denoting by $E_1, \ldots, E_s$ an edge path labeled $f_1, \ldots, f_s$ that connects V_0 to V, the Transfer Equation 3 tells us that the label of V is the subgroup $\left(\mathrm{M}_{f_s,\mathbf{t}}\mathrm{M}_{f_s,\mathbf{s}}^{-1} \cdots \mathrm{M}_{f_1,\mathbf{t}}\mathrm{M}_{f_1,\mathbf{s}}^{-1}\right)\Lambda_0$ of $\mathbb{Z}^d$. In particular, the set of labels of the vertices of $\mathcal{G}$ is exactly $\{\rho(\gamma)\Lambda_0, \gamma \in \mathbb{F}_r\}$.

Proof. Let $\mathscr{C} \in Conj(G_v)$ and let $\Lambda_0 \in \mathscr{C}$. Let us denote by $\Gamma_0 := \mathrm{Stab}_{\mathbb{F}_r}(\Gamma_0)$. Let $(H_i)_{i\in\mathbb{N}^*} \in (\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}})^{\mathbb{N}}$ be the sequence of finitely generated subgroups such that $H_i \cap G_v = \Lambda_0$ for every $i \in \mathbb{N}^*$. For every $i \in \mathbb{N}^*$, let α_i be the saturated $\mathscr{H}$-preaction associated to H_i, and let $\mathcal{G}_i$ be its $\mathscr{H}$-graph. Let β_i be a subpreaction of α_i that corresponds to the same subgroup as α_i and whose $\mathscr{H}$-graph K_i is a finite subgraph of $\mathcal{G}_i$ that contains V_i (legit, because H_i is finitely generated). Let S_i be the graph obtained by forgetting the labels of the vertices of $\mathcal{G}_i$, pointed at the vertex V_i. The graph S_i is the Schreier graph of $\pi(H_i)$ with respect to the generating set $\{a_1, ...a_r\}$ of $\mathbb{F}_r$. As $\pi(H_i)$ is a subgroup of Γ_0, denoting by S_0 the Schreier graph of Γ_0, the graph S_i is in fact a covering of the graph S_0 (for every $i \in \mathbb{N}^*$). As $\pi(H_i)$ belongs to $\mathcal{K}(\Gamma_0)$, one has the following dichotomy:

- either Γ_0 is not finitely generated;
- or Γ_0 is finitely generated and the covering map $S_i \to S_0$ has infinite degree for every $i \in \mathbb{N}^*$.

Let us define F_i as the subgraph of S_i obtained by forgetting the labels of the subgraph K_i og $\mathcal{G}_i$. In both cases, Remark 2.3 provides a covering map $S \to S_0$ such that S contains $(F_i)_{i\in\mathbb{N}^*}$ as disjoint subgraphs and such that the quotient $S/\bigsqcup_{i\in\mathbb{N}^*} F_i$ is a tree. This covering corresponds to an infinite index subgroup $\Gamma \in \mathrm{Sub}_{[\infty]}(\Gamma_0)$. After labeling the vertex V_1 by the subgroup Λ_0 (and all the other vertices of S so that the Transfer Equation 3 is satisfied), we obtain an infinite $\mathscr{H}$-graph $\mathcal{F}$:

- that contains $(K_i)_{i\in\mathbb{N}^*}$ as disjoint subgraphs;
- such that the quotient $\mathcal{F}/\bigsqcup_{i\in\mathbb{N}^*} K_i$ is a tree.

Thus, by Lemma 3.11, there exists a saturated $\mathscr{H}$-preaction γ that extends β_i for every $i \in \mathbb{N}^*$ and whose $\mathscr{H}$-graph $\mathcal{F}$ is infinite. Denoting by L a subgroup of G associated to γ, as the $\mathscr{H}$-graph of L contains a vertex labeled Λ_0, one has $L \cap G_v \in \mathscr{C}$ and as the covering map $S \to S_0$ has infinite degree, one also has $\pi(L) \in \mathrm{Sub}_{[\infty]}(\mathbb{F}_r) \cap \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(L \cap G_v))$, which concludes the proof. □

Hence we obtain the following decomposition of $\mathcal{K}(G)$:

Theorem 6.15. *There exists a G-invariant countable partition*

$$\mathcal{K}(G) = \bigsqcup_{\mathscr{C}\in Conj(G_v)} \mathcal{P}_{\mathscr{C}}$$

into F_σ-subsets of $\mathcal{K}(G)$, and, for every $\mathscr{C} \in Conj(G_v)$, a countable G-invariant open subset $\mathcal{D}_{\mathscr{C}} \subseteq \mathcal{P}_{\mathscr{C}}$ (for the induced topology on $\mathcal{P}_{\mathscr{C}}$) such there exists a dense orbit in $\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$. Moreover, denoting by $\mathscr{C} = \{\rho(\gamma)\Lambda_0, \gamma \in \mathbb{F}_r\}$ for some $\Lambda_0 \leq \mathbb{Z}^d$:

1. *if Λ_0 has finite index in $\mathbb{Z}^d$, then $\mathcal{P}_{\mathscr{C}}$ is a clopen set;*

2. *$\mathcal{P}_{\mathscr{C}}$ is closed iff $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$;*

3. *if $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ either is infinitely generated or has finite index in $\mathbb{F}_r$, then $\mathcal{D}_{\mathscr{C}} = \varnothing$.*

Proof. The existence of a dense orbit in $\mathcal{P}_{\mathscr{C}} \smallsetminus \mathcal{D}_{\mathscr{C}}$ is provided by Lemma 6.14. The fact that $\mathcal{P}_{\mathscr{C}}$ is an F_σ results from Lemma 2.1.

Now let us turn to the proof of the second item. Let $\Lambda_0 \leq \mathbb{Z}^d$ and let $\mathscr{C}$ be the G-conjugacy class of Λ_0. The fact that $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$ is equivalent to the finiteness of the orbit $\{\rho(g)\Lambda_0, g \in G\}$. In particular, $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$, then

$$\mathcal{P}_{\mathscr{C}} = \bigcup_{g \in \mathbb{F}_r} \{H \in \mathcal{K}(G) \mid H \cap G_v = \rho(g)\Lambda_0\}$$

is closed as a finite union of closed sets by Lemma 2.1. Conversely, let us assume that the set $\{\rho(g)\Lambda_0, g \in G\}$ is infinite. In particular, there exists a sequence $(g_n)_{n\in\mathbb{N}} \in \mathbb{F}_r^{\mathbb{N}}$ such that $\rho(g_n)\Lambda_0$ converges to a subgroup $\Lambda \leq G_v$ of rank strictly less than the one of Λ_0. In particular, $\rho(g_n)\Lambda_0 \in \mathcal{P}_{\mathscr{C}}$ for every n, but $\Lambda \notin \mathcal{P}_{\mathscr{C}}$ which implies that $\mathcal{P}_{\mathscr{C}}$ is not closed.

The third item results from Remark 6.13. □

Remark 6.16. Again, we didn't use the cocompactness of the action of G on its Bass-Serre tree. In other words, Theorem 6.15 extends to the case where $r = \infty$.

To conclude, we give an explicit example where the set $\mathcal{D}_{\mathscr{C}}$ is non-empty, and where the conjugation action has no dense orbit for the induced topology on $\mathcal{P}_{\mathscr{C}}$:

Example 6.17. Let P_1 and P_2 be two matrices that generate a free subgroup of $SL_2(\mathbb{Z})$; for instance, $P_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Let G be the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{F}_2$, where, denoting by $\{a_1, a_2\}$ a basis of $\mathbb{F}_2$, the generator a_i acts by multiplication by P_i on $\mathbb{Z}^2$.

Let $\Lambda_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{Z}$. Then, any matrix of $SL_2(\mathbb{Z})$ stabilizing the subgroup Λ_0 of $\mathbb{Z}^2$ is a power of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. As the morphism $\rho : \begin{array}{ccc} \mathbb{F}_2 & \to & SL_2(\mathbb{Z}) \\ a_1 & \mapsto & P_1 \\ a_2 & \mapsto & P_2 \end{array}$ is injective, the stabilizer $\mathrm{Stab}_{\mathbb{F}_2}(\Lambda_0)$ is also infinite cyclic (in fact, it is the subgroup generated by a_1). In particular, its perfect kernel is empty, so denoting by $\mathscr{C} := \mathbb{F}_2 \cdot \Lambda_0$ the G-conjugacy class of Λ_0, one has

$$\mathcal{D}_{\mathscr{C}} = \mathcal{P}_{\mathscr{C}}.$$

Let us show that there is no dense orbit for the conjugation action $G \curvearrowright \mathcal{P}_{\mathscr{C}}$. For any $v \in \mathbb{Z}^2$, let us consider the following invariant subsets

$$U_v = \{H \in \mathcal{P}_{\mathscr{C}} \mid \exists g \in G, g(v, a_1)g^{-1} \in H\}.$$

The set U_v is open as a union of clopen sets:

$$U_v = \bigcup_{g \in G} \{H \in \mathcal{P}_{\mathscr{C}} \mid g(v, a_1)g^{-1} \in H\}.$$

Claim 6.18. *For every $v, w \in \mathbb{Z}^2$ such that $v - w \notin \Lambda_0$, one has $U_v \cap U_w = \varnothing$.*

As the quotient $\mathbb{Z}^2/\Lambda_0$ is infinite, this will imply the existence of an infinite countable set of pairwise disjoint open subsets, contradicting the existence of a dense orbit.

Proof of the claim. By contraposition, let $v, w \in \mathbb{Z}^2$ such that $U_v \cap U_w \neq \varnothing$. Let $H \in \mathcal{P}_{\mathscr{C}}$ and $g \in G$ such that

- $(v, a_1) \in H$;
- $g(w, a_1)g^{-1} \in H$.

As $H \in \mathcal{P}_{\mathscr{C}}$, there exists $\gamma \in \mathbb{F}_2$ such that $H \cap \mathbb{Z}^2 = \rho(\gamma)\Lambda_0$. Thus, $\mathrm{Stab}_{\mathbb{F}_2}(H \cap \mathbb{Z}^2) = \gamma\langle a_1\rangle\gamma^{-1}$. Denoting by $g = (u_0, \gamma_0)$, one deduces that $a_1, \gamma_0 a_1 \gamma_0^{-1} \in \gamma\langle a_1\rangle\gamma^{-1}$. This forces γ and γ_0 to belong to $\langle a_1\rangle$. In particular, $H \cap \mathbb{Z}^2 = \Lambda_0$. Let $k \in \mathbb{Z}$ such that $\gamma_0 = a_1^k$. One has

$$\begin{aligned}
g(w, a_1)g^{-1}(v, a_1)^{-1} &= \left(u_0, a_1^k\right)(w, a_1)\left(u_0, a_1^k\right)^{-1}(v, a_1)^{-1} \\
&= \left(u_0 + \rho(a_1)^k w, a_1^{k+1}\right)\left(-\rho(a_1)^{-k}u_0, a_1^{-k}\right)(v, a_1)^{-1} \\
&= \left(u_0 + \rho(a_1)^k w - \rho(a_1)u_0, a_1\right)\left(-\rho(a_1)^{-1}v, a_1^{-1}\right) \\
&= \left(u_0 - P_1 u_0 + P_1^k w - v, 1\right) \\
&\in \left(H \cap \mathbb{Z}^2\right) \times \{1\} \\
&= \Lambda_0 \times \{1\}.
\end{aligned}$$

Thus, $u_0 - P_1 u_0 + P_1^k w - v \in \Lambda_0$. Notice that $(I_2 - P_1)\mathbb{Z}^2 \subseteq \Lambda_0$. Thus,

$$u_0 - P_1 u_0 \in \Lambda_0,$$

and

$$\begin{aligned}
P_1^k w &= w + \sum_{i=0}^{k-1} P_1^i (P_1 w - w) \\
&\in w + \Lambda_0,
\end{aligned}$$

which implies that $w - v \in \Lambda_0$. $\square$

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