Bass - Serre theory
A journey into the forest

Plan of the lectures:

1st lecture

0/ Motivations

I/ Amalgamated free products

1) Normal forms
2) The Bass - Serre tree of an amalgamated free product
3) Examples and applications

II/ Structure of a group acting on a tree

1) Graphs of groups
2) Fundamental group of a graph of groups

2nd lecture

3) Reduced words
4) Universal cover of a graph of groups
5) Structure theorem

III/ Some applications

1) Kurosh's theorem
2) Finite graphs of finite groups
3) Grushko's theorem

References : - Arbres, amalgames, SL_2, J.-P. Serre
- Lectures given by Lymer at the summer school "Geometric group theory without boundary" (July 2021)
- Topological methods in groups theory, Scott - Wall

0 / Motivations

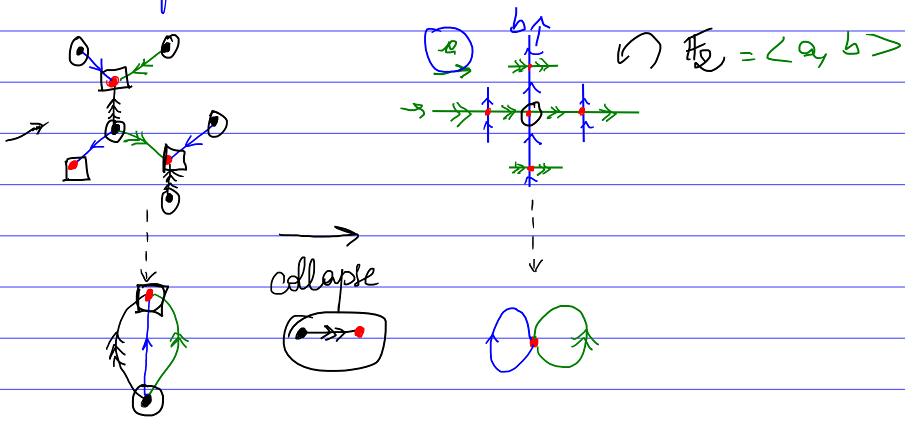
→ Studying groups via their actions on geometric spaces

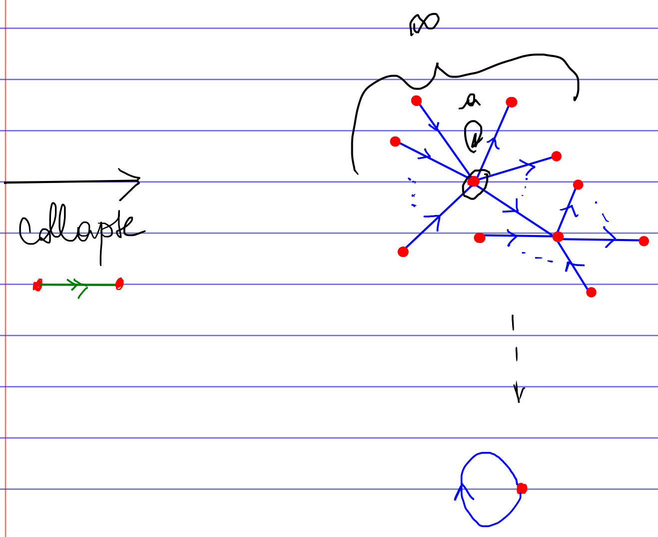
1-dimensional objects: trees!

Th: A group is free iff it acts freely on a tree.

Cor (Nielsen-Schreier): A subgroup of a free group is free.

Ex:



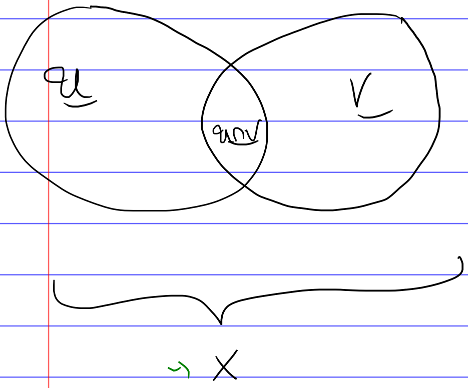


Q : What happens if the action is not free?

⇝ Bass Serre theory tells us that only certain groups can act on trees with prescribed vertex and edge stabilizers!

Why are trees interesting?

⇝ Van Kampen's theorem!

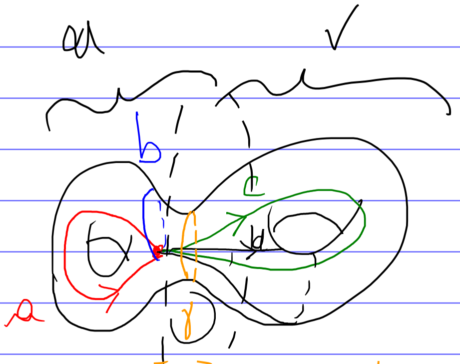


$$\Rightarrow \pi_1(X) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U}\cap V)} \pi_1(V)$$

Def: A_1, A_2, B groups, $i_1 : B \to A_1$, $i_2 : B \to A_2$ morphisms.

$$A_1 *_B A_2 = \langle A_1, A_2 \mid R_{A_1}, R_{A_2}, i_1(b) i_2(b)^{-1} \ \forall b \in B \rangle$$

Ex: Σ_2

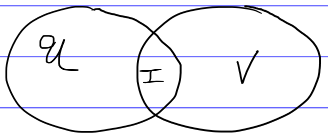


$= [a,b] = aba^{-1}b^{-1}$
$= [c,d]$

$$\pi_1(\Sigma_2) = \underset{\substack{\| \\ \langle a,b\rangle \\ \cong \\ F_2}}{\pi_1(\mathcal{U})} \underset{\substack{\pi_1(\mathcal{U}\cap V) \\ \| \\ \langle \gamma \rangle}}{*} \underset{\substack{\| \\ \langle c,d\rangle \\ \cong \\ F_2}}{\pi_1(V)}$$

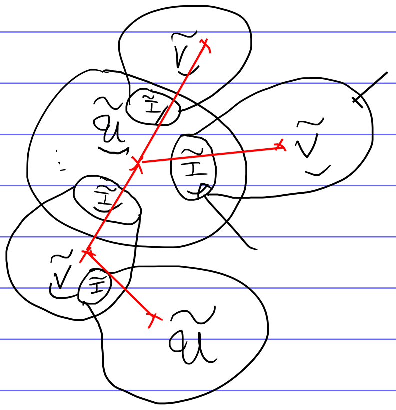
$$= \langle a, b, c, d \mid [a,b] = [c,d] \rangle$$

$X =$



with $\pi_1(I) \hookrightarrow \pi_1(\mathcal{U}), \pi_1(V)$

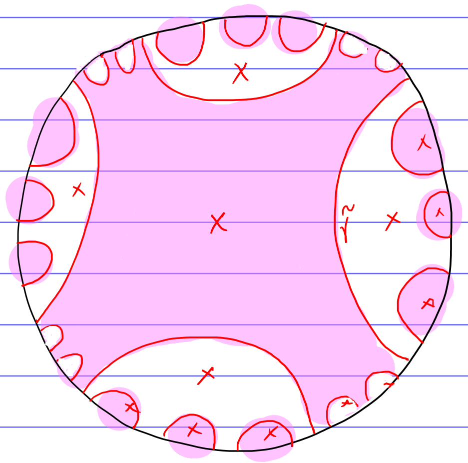
$\Rightarrow \tilde{X} =$

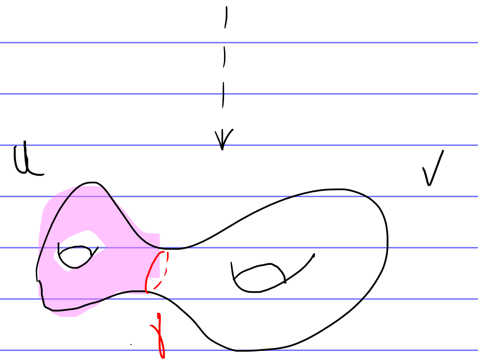


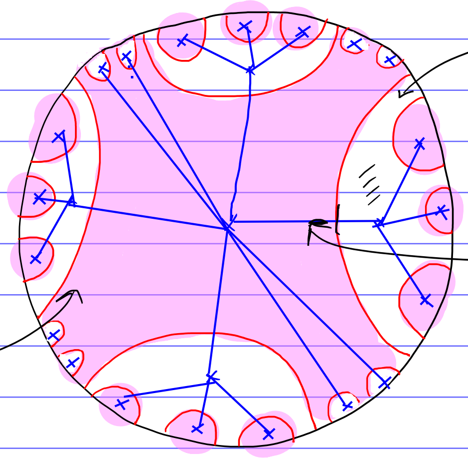
"collapsing $\tilde{\mathcal{U}}$ and $\tilde{V}$ leads to a tree on which $\pi_1(X)$ acts"

Ex:

$\mathbb{H}^2$





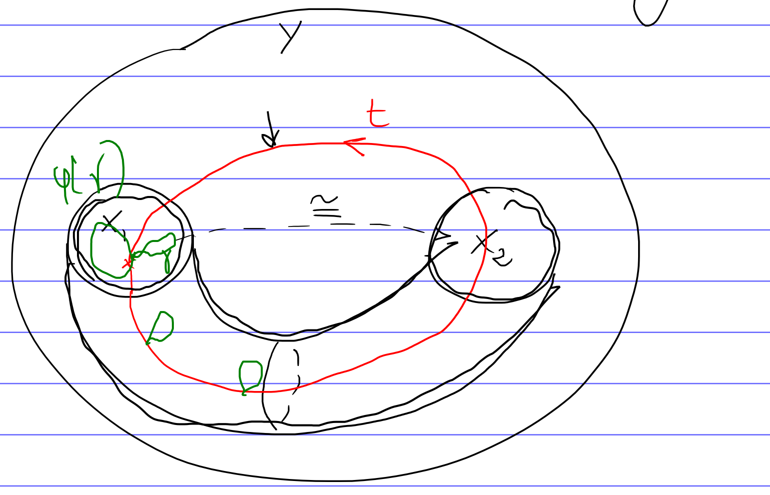


$\mathrm{Stab}(\cdot) = \pi_1(V)$

$\mathrm{Stab}(\times\!\!-\!\!\!-\!\!\times)$
$= \langle \gamma \rangle$

$\mathrm{Stab}(\cdot)$
$= \pi_1(\mathcal{U})$

What if the curve is non separating?

$X =$ 

$$\pi_1(X) = \left\langle \pi_1(Y), t \;\middle|\; R_{\pi_1(Y)},\ t\gamma t^{-1} = \varphi(\gamma)\ \forall \gamma \in \pi_1(X_1) \right\rangle$$

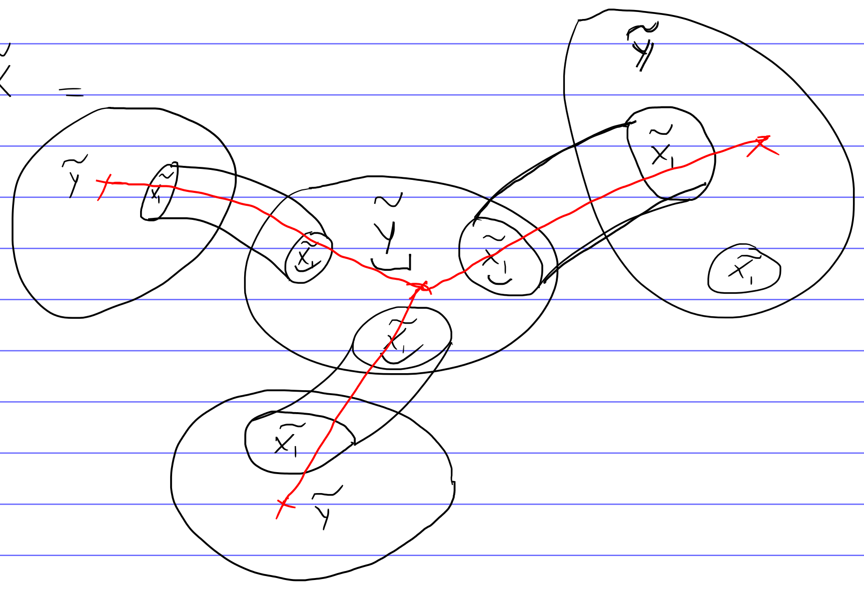
$$\varphi : \pi_1(X_1) \circlearrowleft$$

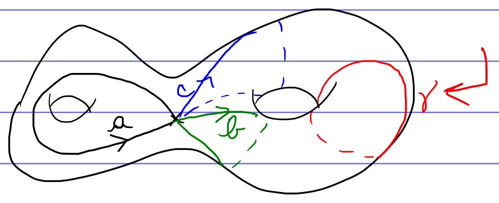
Def: G group, $H, K \leq G$
$\varphi : H \to K$ isomorphism

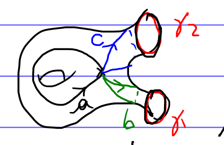
$$G *_\varphi = \left\langle G, t \;\middle|\; R_G,\ tht^{-1} = \varphi(h)\ \forall h \in H \right\rangle$$

"Conjugate the subgroups of G that are isomorphic"

$\pi_1(X_1) \hookrightarrow \pi_1(X)$

$\tilde{X} =$ 

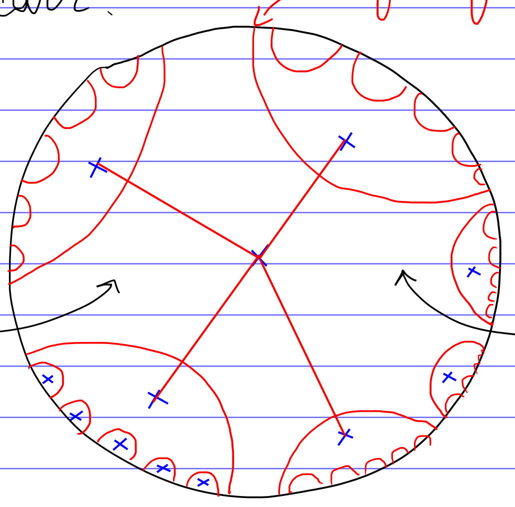
Ex: 

$\pi_1(\Sigma_2) = \pi_1\Big($  $\Big) *_{\langle\gamma\rangle}$

$= \langle a, b, c, t \mid tbt^{-1} = c \rangle.$

Universal cover:

lifts of γ



Stab(·) $= \pi_1(\text{◎})$

Stab(x—x) $= \langle\gamma\rangle$

"Philosophy of Bass-Serre theory": groups acting nicely on trees SPLIT as {amalgamated free products / HNN extensions} of vertex stabilizers over edge stabilizers.

I/ Amalgamated free products

Def: A_1, A_2, B groups, $\begin{cases} B \overset{i_1}{\hookrightarrow} A_1 \\ B \overset{i_2}{\hookrightarrow} A_2 \end{cases}$ injections.

$$A_1 *_B A_2 = A_1 * A_2 / \langle\langle i_1(b)\, i_2(b)^{-1} \ \forall b \in B \rangle\rangle$$

1) Normal forms

$\mathbb{F}_2 = \langle a, b \rangle$
$x = a^{k_1} b^{l_1} a^{k_2} \_\_\_\_$

$G = A_1 *_B A_2$ $\qquad$ S_i set of representatives of $B \backslash A_i$.

Not: $f_i : A_i \longrightarrow G$
$f : B \longrightarrow G$

⚠ It turns out that these morphisms are injective, but this is not trivial!

Def: A reduced word of type $(\varepsilon_1, \ldots, \varepsilon_k)$ is a $(k+1)$-uple $x = (b; s_1, \ldots, s_k)$, where

- $\varepsilon_i \in \{1, 2\}$ $\forall i$;
- $\varepsilon_i \neq \varepsilon_{i+1}$;
- $b \in B$;
- $s_i \in S_i \setminus \{1\}$.

$$A_i = \coprod_{s \in S_i} B s$$

The element of G associated to x is

$$\beta(x) := f(b)\, f_{\varepsilon_1}(s_1) \ldots f_{\varepsilon_k}(s_k).$$

Theorem: Let $g \in G$. There exists a unique reduced word m such that $g = \beta(m)$

Proof: The non trivial part is the uniqueness.

$X = \{\text{reduced words}\}$

$Y_1 = \{\text{Reduced words } (1; \text{—}) \text{ of type } (2,1,-)\}$

$Y_2 = \{\text{Reduced words } (1; \text{—}) \text{ of type } (1,2,-)\}$

Goal: Build $G \curvearrowright X$.

Obs: $X \simeq Y_i \times A_i$

$$\begin{cases} (b, s_i, x_1, x_2, -) \leftrightarrow (1, x_1, x_2, \text{—}), & \underset{\substack{\Downarrow\\ B}}{b}\ \underbrace{s_i}_{\neq 1}, \ s_i \in A_{i+1} \\ (b, x_1, x_2, \text{—}) \leftrightarrow (1, x_1, x_2, \text{—}), & \underset{B}{b} \end{cases}$$

$A_i \curvearrowright Y_i \times A_i$ by $a \cdot (y, a') = (y, aa')$

$\Rightarrow A_i \curvearrowright X$ and the induced B action

is $b \cdot (b'; x_1, \text{—}) = (bb', x_1, -)$

which does not depend on i.

$\Rightarrow$ Defines a unique action $G \curvearrowright X$.

Obs: $\beta(\overset{\text{reduced}}{\hat{m}}) \cdot (1; \emptyset) = m$ (induction)

$\Rightarrow \beta$ is injective ■

Corollary: $(i_1, \dots, i_k) \in \{1,2\}^k$ st $i_j \neq i_{j+1}$ $\forall j$. Then:

$$b \underbrace{a_{i_1}}_{\in A_{i_1} \setminus B} \overbrace{a_{i_2}}^{\in A_{i_2} \setminus B} \text{—} \underbrace{a_{i_k}}_{\in A_{i_k} \setminus B} \neq 1. \quad \left(\text{except when } b=1 \text{ and } k=0\right)$$

Corollary: $A_i \overset{f_i}{\hookrightarrow} G$.

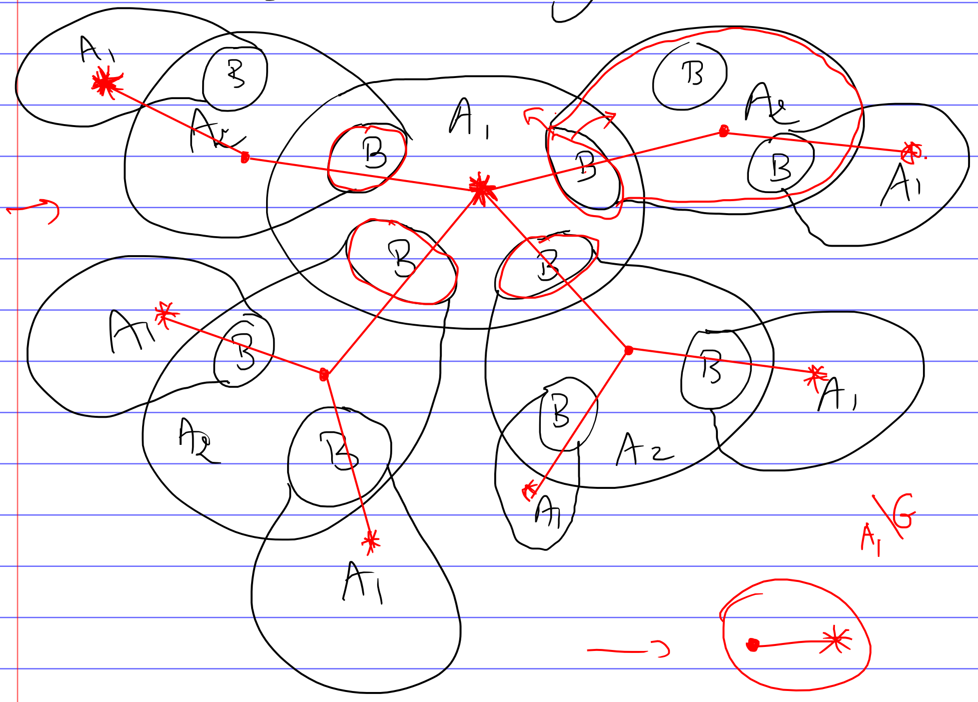
Corollary: A finite order element is conjugate to an element of A_1 or of A_2.

Proof: $g \in G$ torsion element. Up to conjugating, one can assume that g is of type $(i_1, \dots, i_k)$ with $i_k \neq i_1$.

$\Rightarrow$ g^n of type $(\underbrace{i_1, \dots, i_k, i_1, \dots, i_k, \dots, i_1, \dots, i_k}_{n \text{ times}})$, hence $\neq 1$.

2) The Bass-Serre tree of an amalgamated free product

Idea: Cayley graph of $A_1 *_B A_2$:



$\nexists$ circuit $\Leftrightarrow$ theorem on normal forms.

Theorem 1: Let $G \curvearrowright X$ graph and let $P \xrightarrow{y} Q$ be a fundamental domain.

Not: $G_P = \mathrm{Stab}\, P$, $G_Q = \mathrm{Stab}\, Q$, $G_y = \mathrm{Stab}\, y$.

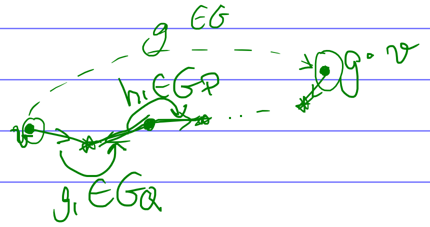
TFAE:

- X is a tree;
- The homomorphism $G_P *_{G_y} G_Q \to G$ induced by $\left\{ \begin{array}{l} G_P \hookrightarrow G \\ G_y \hookrightarrow G \\ G_Q \hookrightarrow G \end{array} \right.$ is an isomorphism. $G_P \hookleftarrow G_y \hookrightarrow G_Q$

Proof:

1) X is connected $\Leftrightarrow$ G_P, G_Q generate G.

$\boxed{\Rightarrow}$

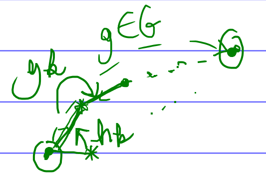


$$\Rightarrow g \cdot v = \underbrace{g_1 h_1 \cdots h_{k-1} g_k}_{: h \,\in\, \langle G_P, G_Q \rangle} v$$

$$\Rightarrow h^{-1} g v = v \Rightarrow h^{-1} g \in G_P$$

$$\Rightarrow g \in \langle G_P, G_Q \rangle .$$

$\boxed{\Leftarrow}$



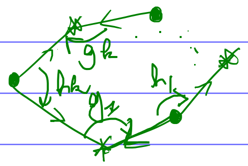
, with $g = g_1 h_1 \cdots g_k h_k$, $g_i \in G_Q$, $h_i \in G_P$

∎

2) X has no circuit $\Leftrightarrow$ $G_P *_{G_y} G_Q \hookrightarrow G$.

X has a circuit

$\Leftrightarrow \exists$



$\Leftrightarrow \exists \begin{cases} g_i \in G_Q \setminus G_y \\ h_i \in G_P \setminus G_y \end{cases}$, $g_1 h_1 \cdots g_k h_k v = v$

i.e. $g_1 h_1 \cdots g_k h_k \in G_y$

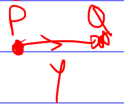
$\Leftrightarrow$ $G_P *_{G_y} G_Q \not\hookrightarrow G$ (normal forms) ∎

∎

<u>Corollary</u> $G \curvearrowright T$ tree with quotient $\simeq P \xrightarrow{y} Q \subseteq T$

$$\Rightarrow G \simeq G_P *_{G_y} G_Q .$$

Theorem: Let $G = A_1 *_B A_2$ be an amalgamated free product. There exists a (unique up to unique isomorphism) tree T on which G acts with fundamental domain

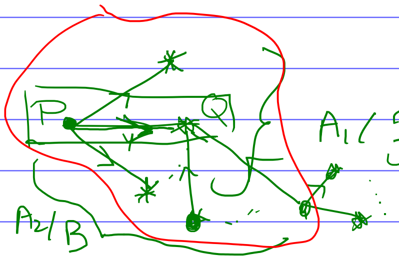


or

- $\mathrm{Stab}\ P = A_1$
- $\mathrm{Stab}\ Q = A_2$
- $\mathrm{Stab}\ y = B$.

Proof:

Uniqueness:



T graph. vertices: $\underbrace{G/A_1}_{\text{type } \bullet} \sqcup \underbrace{G/A_2}_{\text{type } *}$

Edges: $G/B \sqcup \overline{G/B}$

Existence: T is a tree by the previous theorem.

$\blacksquare$

3) Examples and applications

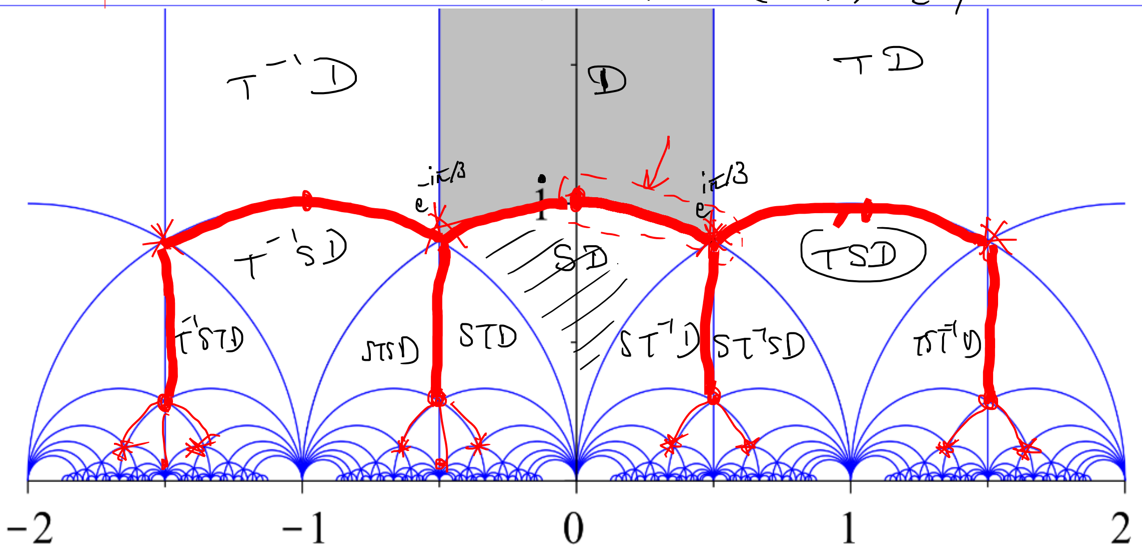
a) One example: $SL_2(\mathbb{Z})$

$$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$$

via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$

fund. dom. D for the action:

$$\left\{\begin{array}{ll} T=\begin{pmatrix}1&1\\0&1\end{pmatrix} & \begin{pmatrix}\mathbb{H}\to\mathbb{H}\\ z\mapsto z+1\end{pmatrix}\\ S=\begin{pmatrix}0&-1\\1&0\end{pmatrix} & \begin{pmatrix}\mathbb{H}\to\mathbb{H}\\ z\mapsto -\frac{1}{z}\end{pmatrix}\end{array}\right.$$



Bass-Serre theory $\Rightarrow$ $SL_2(\mathbb{Z}) = \mathrm{Stab}_i \ast_{\mathrm{Stab}_i \cap \mathrm{Stab}_{e^{i\pi/3}}} \mathrm{Stab}_{e^{i\pi/3}}$

- $\mathrm{Stab}\, i = \left\{ \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \frac{ai+b}{ci+d} = i \right\}$

$$= \left\{ \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix} \right\}$$

$$= \left\langle \begin{pmatrix}0&1\\-1&0\end{pmatrix} \right\rangle$$

$$\simeq \mathbb{Z}/4\mathbb{Z}$$

- $\mathrm{Stab}\, e^{i\pi/3} = \left\{ \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \underbrace{\frac{ae^{i\pi/3}+b}{ce^{i\pi/3}+d} = e^{i\pi/3}}_{\Updownarrow \; ce^{2i\pi/3} + (d-a)e^{i\pi/3} - b = 0} \right\}$

$$= \left\{ \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2[\mathbb{Z}] \;\middle|\; d-a = -b = c \right\}$$

$\left(e^{i\pi/3}, Q(x) = x^2+x+1\right)$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$= \left\langle \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$$\simeq \mathbb{Z}/6\mathbb{Z}$$

- $\mathrm{Stab}\, i \cap \mathrm{Stab}\, e^{i\pi/3} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \simeq \mathbb{Z}/2\mathbb{Z}$

$$\rightsquigarrow \boxed{\begin{aligned} SL_2(\mathbb{Z}) &= \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \\ &= \langle a, b \mid a^6, b^4, a^3 b^{-2} \rangle. \end{aligned}}$$

b) Some consequences

<u>Prop</u>: $\Gamma \leq A_1 *_B A_2$ st $\Gamma \setminus \{1\}$ does not meet any conjugate of G_1 or G_2. Then Γ is free.

<u>Proof</u>: Γ acts freely on the Bass-Serre tree of $A_1 *_B A_2$. ∎

<u>Th</u>: Any bounded subgroup of $A_1 *_B A_2$ is contained in a conjugate of A_1 or A_2.

<u>Proof</u>: Γ bounded subgroup, v vertex of the Bass-Serre tree T
$\Gamma \cdot v$ is bounded

$\Rightarrow$ $\mathrm{Conv}(\Gamma \cdot v)$ is a finite subtree of T, on which Γ acts.

$\Rightarrow$ $\exists$ a global fixed point in $\mathrm{Conv}(\Gamma \cdot v)$

$\Rightarrow$ Γ fixes a vertex of T. ∎

Cor: Any finite subgroup of $A_1 *_B A_2$ is contained in a conjugate of A_1 or A_2.

Proof: Any finite subgroup is bounded. ∎

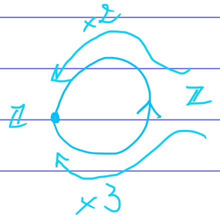
II / Structure of a group acting on a tree

1) Graphs of groups

Def: A graph of groups (G,Y) is the data of:

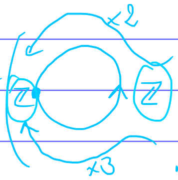
- Y an oriented graph;
- $\forall v \in \mathcal{V}(Y)$, G_v a group
  $\forall y \in \mathcal{E}(Y)$, $G_y = G_{\bar{y}}$ a group and $i_y : G_y \hookrightarrow G_{t(y)}$

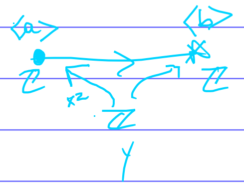
Ex:



Not: • if $a \in G_y$, $i_y(a) := a^y$

• $F(G,Y) = \mathop{\ast}_{v \in V(Y)} G_v * \langle y, y \in \mathcal{E}(Y) \rangle \Big/ \langle\langle \bar{y} = y^{-1},\ y a^y y^{-1} = a^{\bar{y}} \rangle\rangle$

Ex: F() $= \langle a, y \mid y a^2 y^{-1} = a^3 \rangle$

F() $= \langle a, b, y \mid y a^2 y^{-1} = b^3 \rangle$

$y_i \xrightarrow{} P_i \xrightarrow{} y_{i+1}$

Def: $(y_1, \ldots, y_k)$ an edge path of Y (i.e. $o(y_{i+1}) = t(y_i) := P_i$ $\forall i$). A word of type $c = (y_1, \ldots, y_k)$ is a couple (c, μ) where $\mu = (r_0, \ldots, r_k)$, with $r_i \in G_{P_i}$ $\forall i$.

The associated element of $F(G, Y)$ is

$$|c, \mu| = r_0 \, y_1 \, r_1 \, y_2 \ldots y_k \, r_k .$$

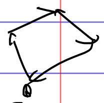
$P_0 \ni r_0 \xrightarrow{y_1} r_1 \xrightarrow{y_2} r_2 \xrightarrow{y_3} r_3 \rightsquigarrow r_0 y_1 \cancel{r_1} y_2 r_2 y_3 r_3$

2) Fundamental group of a graph of groups

Two equivalent definitions of the fundamental group of a graph of groups (G, Y):

a) Choose a base point $P_0 \in V(Y)$.

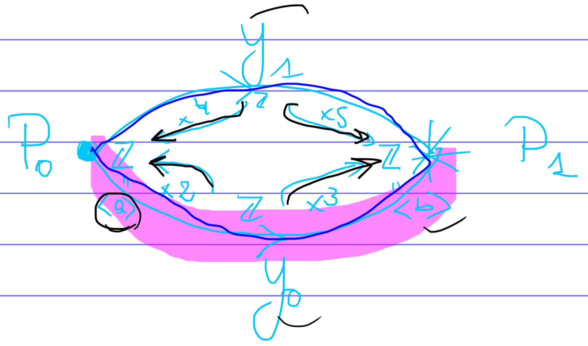
Define $\boxed{\pi_1(G, Y, P_0) = \langle\, |c, \mu|,\ (c, \mu) \text{ word of type } (y_1, \ldots, y_k) \text{ with } o(y_1) = t(y_k) = P_0 \,\rangle}$


P_0

b) Choose a spanning tree T of Y.

Define $\boxed{\pi_1(G, Y, T) = F(G, Y) \big/ \langle\langle y,\ y \in T \rangle\rangle}$

Ex :



$$F(G,Y) = \langle a, b, y_0, y_1 \mid y_0 a^2 y_0^{-1} = b^3,\ y_1 a^4 y_1^{-1} = b^5 \rangle$$

$$\vee\!\mid$$

$$\pi_1(G,Y,P_0) = \langle a,\ y_i b^k y_j^{-1},\ k \in \mathbb{Z} \rangle$$

$y_0 y_1^{-1}$ $\quad y_0 b$ $\quad y_0^{-1}$

$\uparrow \qquad \uparrow$

$\bar{y}_1 \qquad b$

$$\pi_1(G,Y,T) = F(G,Y) / \langle\langle y_0 \rangle\rangle$$

$$= \langle a, b, y_1 \mid a^2 = b^3,\ y_1 a^4 y_1^{-1} = b^5 \rangle$$

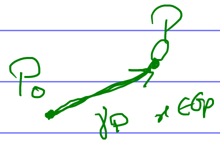
Theorem: The restriction of the projection $F(G,Y) \xrightarrow{\pi} \pi_1(G,Y,T)$ induces an isomorphism between $\pi_1(G,Y,P_0)$ and $\pi_1(G,Y,T)$.

Proof: We will define the inverse map

$$\pi_1(G,Y,T) \longrightarrow \pi_1(G,Y,P_0).$$

- $P \in V(Y) \rightsquigarrow \exists!$ edge path $y_0^{(P)}, \dots, y_{k_P}^{(P)}$ in T, $\begin{cases} o(y_0) = P_0 \\ t(y_k) = P \end{cases}$

Not : $\gamma_P := y_0^{(P)} \cdots y_{k_P}^{(P)} \in F(G,Y)$

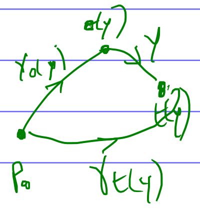


Let

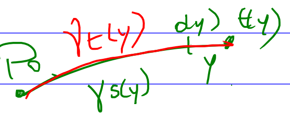
$$\pi_1(G,Y,T) \xrightarrow{\ \varphi\ } \pi_1(G,Y,P_0)$$

$$x \in G_P \longmapsto \gamma_P\, x\, \gamma_P^{-1}$$

$$y \in E(Y) \longmapsto \gamma_{o(y)}\, y\, \gamma_{t(y)}^{-1}$$



It is well-defined:



$\gamma_{s(y)}\, y = \gamma_{t(y)}$

- $y \in T \Rightarrow \varphi(y) = 1$.

and for every $a \in G_y$:

$$\underset{G_{t(y)}}{\varphi(a^y)} = \gamma_{t(y)}\, a^y\, \gamma_{t(y)}^{-1}$$
$$= \gamma_{s(y)}\, \underbrace{y\, a^y\, y^{-1}}_{= a^{\bar{y}}}\, \gamma_{s(y)}^{-1}$$
$$= \gamma_{s(y)}\, a^{\bar{y}}\, \gamma_{s(y)}^{-1}$$
$$= \varphi(a^{\bar{y}})$$

- $y \notin T \Rightarrow \forall a \in G_y$,

$$\varphi(y)\varphi(a^y)\varphi(y)^{-1} = \left(\gamma_{s(y)}\, y\, \cancel{\gamma_{t(y)}^{-1}}\right)\left(\cancel{\gamma_{t(y)}}\, a^y\, \cancel{\gamma_{t(y)}^{-1}}\right)\left(\cancel{\gamma_{t(y)}}\, y^{-1}\gamma_{s(y)}^{-1}\right)$$
$$= \gamma_{s(y)}\, \underbrace{y\, a^y\, y^{-1}}_{a^{\bar{y}}}\, \gamma_{s(y)}^{-1}$$
$$= \gamma_{s(y)}\, a^{\bar{y}}\, \gamma_{s(y)}^{-1}$$
$$= \varphi(a^{\bar{y}}).$$

- $$\pi \circ \varphi(\underset{\in G_P}{x}) = \underbrace{\pi(\gamma_P)}_{=1}\, x\, \pi(\gamma_P)^{-1}$$
$$= x$$

$$\pi \circ \varphi(\underset{\in \mathcal{E}(Y)}{y}) = \underbrace{\pi(\gamma_{o(y)})}_{=1}\, y\, \underbrace{\pi(\gamma_{t(y)})^{-1}}_{=1}$$
$$= y.$$

- $(c, \mu) = ((y_0, \dots, y_n), (r_0, \dots, r_n))$ with $t(y_n) = o(y_0) = P_0$, $P_i := t(y_i)$

$|c, \mu| = r_0 y_0 r_1 y_1 \cdots r_n y_n$

& $\varphi \circ \pi(|c,\mu|) = \left(\gamma_{P_0} r_0 \cancel{\gamma_{P_0}^{-1}}\right)\left(\cancel{\gamma_{P_0}}\, y_0\, \cancel{\gamma_{P_1}^{-1}}\right)\left(\cancel{\gamma_{P_1}}\, r_1\, \gamma_{P_1}^{-1}\right)\left(\gamma_{P_1}\, y_2\, \gamma_{P_2}^{-1}\right)\cdots$

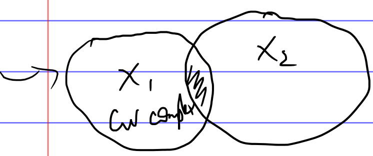
$\underset{(\gamma_{P_0} = 1)}{=} |c, \mu|$. ▨

Corollary: - $\pi_1(G, Y, P_0)$ doesn't depend on the choice of P_0;

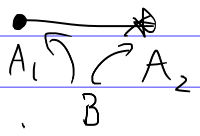
- $\pi_1(G, Y, T)$ doesn't depend on the choice of T.

End of the first lecture

Why "fundamental group"?



$\hookrightarrow$



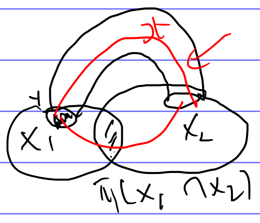
$\pi_1(X_i) = A_i$

$\pi_1(B) = \pi_1(X_1 \cap X_2)$

Bass (1993)

Scott Wall

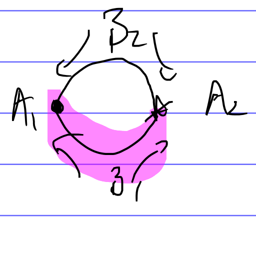
$\pi_1(y) = B_2$



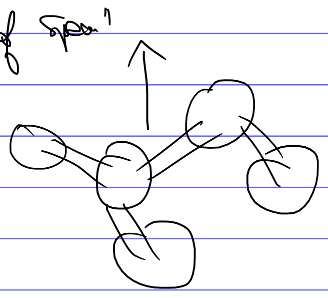
$\pi_1(X_i) = A_i$

$\pi_1(X_1 \cap X_2) = B_1$

$\hookrightarrow$



"graph of spaces"



$\hookrightarrow$

Bass Serre ~~tree~~

Reminder

→ Graph of groups: · graph Y

· $\forall v \in V(Y)$, a group G_v

$y \in \mathcal{E}(Y)$, a group G_y s.t. $G_{\bar{y}} = G_y$

· injections

$$\begin{tikzcd} G_{o(y)} \arrow[rr, "y"] & & G_{t(y)} \\ & G_y \arrow[ul, "\iota_{\bar{y}}"] \arrow[ur, "\iota_y"'] & \end{tikzcd}$$

Not: $G_y \hookrightarrow G_{t(y)}$

$a \mapsto a^y$

→ word (c,μ): · edge path $c = (y_1, \dots, y_n)$ $\quad t(y_i) = o(y_{i+1})$ $\forall i$ $=: P_i$

$P_0 = o(y_1)$

· $\mu = (r_0, \dots, r_n)$ $\quad r_i \in G_{P_i}$ $\forall i$

→ The definition of the fundamental group of a graph of groups:

$$F(G,Y) := \underbrace{\ast_{v \in V(Y)} G_v}\ \underbrace{\ast\, \mathcal{E}(Y)} \Big/ \left\langle\!\left\langle \bar{y}y \ \ \forall y \in \mathcal{E}(Y),\ y a^y y^{-1} (a^{\bar{y}})^{-1} \right\rangle\!\right\rangle$$

Element of $F(G,Y)$ associated to the word (c,μ):

$|c,\mu| := r_0 y_1 r_1 y_2 \dots y_n r_n \in F(G,Y)$

1st definition	2nd definition
Choose a base point $P_0 \in V(Y)$	Choose a spanning tree T of Y
$\pi_1(G,Y,P_0) = \{ \lvert c,\mu \rvert \mid c \text{ cycle based at } P_0 \}$	$\pi_1(G,Y,T) = \underbrace{F(G,Y) \big/ \langle\langle y,\ y \in \mathcal{E}(T) \rangle\rangle}_{\text{iteration of } \left\{ \begin{array}{l} \text{amalgamated free products} \\ \text{HNN extensions} \end{array} \right. \text{of the vertex groups over the edge groups}}$
$\leq F(G,Y)$	

Thm: $\pi\big|_{\pi_1(G,Y,P_0)} : \pi_1(G,Y,P_0) \xrightarrow{\sim} \pi_1(G,Y,T)$ is an isomorphism

Cor: · $\pi_1(G,Y,P_0)$ doesn't depend on the choice of P_0.

· $\pi_1(G,Y,T)$ doesn't depend on the choice of T.

3) Reduced words

(c, μ) word of type c : $\cdot$ c edge path $y_1 - y_n$ based at P_0 $\quad (o(y_1) = P_0)$

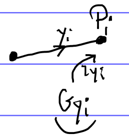
$\cdot$ $\mu = (r_0, \dots, r_n)$ $\quad P_i = t(y_i)$

$r_i \in P_i$

Def: (c, μ) is reduced if :

- either $n = 0$ and $r_0 \neq 1$ $\qquad (c,\mu) = (\emptyset, \underset{\neq 1}{r_0})$
- or $n > 0$ and $y_i = \overline{y_{i+1}} \Rightarrow r_i \notin G_{y_i}^{y_i} \; (\leq G_{P_i})$

Not :

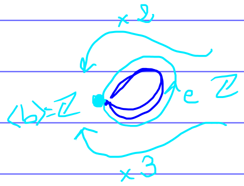


$G_{y_i} \xrightarrow{\iota_{y_i}} G_{P_i}$

$a \longmapsto a^{y_i}$

$\underline{G_{y_i}^{y_i} := \iota_{y_i}(G_{y_i})}$

Ex:



$F(G,Y) = \langle b, t \mid t b^2 t^{-1} = b^3 \rangle$

- $(c, \mu) = \left((e, \bar{e}), \left(b^{-3}, \underbrace{b^3}_{\notin 2\mathbb{Z}}, 1 \right) \right)$ is reduced

$|(c,\mu)| = b^{-3} t b^3 t^{-1}$

$\neq 1$ in $F(G,Y)$

- $(c, \mu) = \left((e, \bar{e}), \left(b^{-3}, \underbrace{b^2}_{\in 2\mathbb{Z}}, 1 \right) \right)$ is not reduced

$|(c,\mu)| = b^{-3} t b^2 t^{-1}$

$= 1$ in $F(G,Y)$.

Theorem: If (c, μ) is reduced, then $|c, \mu| \neq 1$ in $F(G, Y)$.

Cor 1: For every vertex P of Y, $G_P \hookrightarrow F(G, Y)$

Proof: For any $x \in G_P \setminus \{1\}$, the word $(\emptyset, \underset{\neq 1}{x})$ is reduced

$\underset{\text{thm}}{\Rightarrow} |\emptyset, x| = x \neq 1$ in $F(G, Y)$ $\blacksquare$

Cor 2: If $\underset{(= n)}{\ell(c)} \geq 1$, and (c, μ) is reduced, then $|c, \mu| \notin G_{P_0}$, $P_0 = o(y_1)$

Proof Otherwise $\exists\, x \in G_{P_0}$ s.t.

$$\underbrace{|c, \underbrace{(x^{-1} r_0, r_1, \dots, r_n)}_{\text{length } n \geq 1}|}_{\text{reduced}} = 1$$

$|c, \mu| = x \in G_{P_0}$

$x^{-1} r_0 y_1 r_1 \dots y_n = 1$

$\shortparallel$

$|(y_1, \dots, y_n), (x^{-1} r_0, r_1, \dots, y_n)|$

↯ $\blacksquare$

Cor 3: Let T be a spanning tree of Y. If (c, μ) is reduced and c is a closed path, then $|c, \mu| \neq 1$ in $\pi_1(G, Y, T)$

Proof: $|c, \mu| \neq 1$ in $\pi_1(G, Y, P_0)$

$\cap$

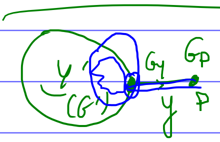
$F(G, Y)$

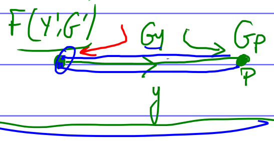
$\Rightarrow |c, \mu| \neq 1$ in $\pi_1(G, Y, T)$. $\blacksquare$

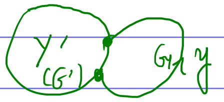
Sketch of the proof of the theorem:

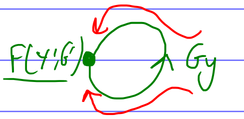
(c, μ) reduced word.

- Induction on the number of edges of Y:

either $\underset{(G)}{Y} =$ 

$\rightsquigarrow \underset{(H)}{W} =$ 

or $\underset{(G)}{Y} =$ 

$\rightsquigarrow \underset{(H)}{W} =$ 

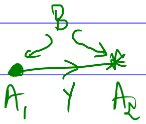
$\hookrightarrow$: induction hypothesis applied on Y' + $\boxed{\text{Cor. 1}}$

- $F(G,Y) \simeq F(H,W)$

- A reduced word (c,μ) in (Y,G) leads to a reduced word (c',μ') in (W,H) (induction hypothesis on Y' + $\boxed{\text{Cor. 2}}$)

- So by the base case of the induction $|c',\mu'| \neq 1 \Rightarrow |c,\mu| \neq 1$ in $F(G,Y)$.

- It remains to treat the base case:

$\hookrightarrow$ either 

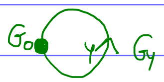
: normal in amalgamated free products.

$$|c,\mu| = \underset{\underset{A_1}{\cap}}{a_1^1}\; y\; \underset{\underset{A_2 \setminus B}{\cap}}{a_2^1}\; \bar{y}\; \underset{\underset{A_1}{\cap}}{a_1^2} \cdots$$

$\xrightarrow{/\langle\langle y\rangle\rangle}$ reduced element in $A_1 *_B A_2$, so $\neq 1$

$\Rightarrow |c,\mu| \neq 1$

$\Rightarrow$ together with the induction step, shows that the theorem is true for trees of groups.

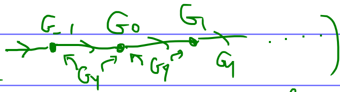
↳ on 

$R_0 := \ker\left(F(G,Y) \to \mathbb{Z},\ G_0 \mapsto 0,\ y \mapsto 1 \right) = \langle\langle G_0 \rangle\rangle$

$\rightsquigarrow R_0 = \langle y^n G_0 y^{-n},\ n \in \mathbb{Z} \rangle$

$= \underset{n\in\mathbb{Z}}{*_{G_y}} G_n$, $\underline{G_n = y^n G_0 y^{-n}}$

where $G_{n-1} \leftarrow G_y \rightarrow G_n$, $\quad y^{n-1}\bar{a}\,y^{-(n-1)} \leftarrow a \mapsto y^n a^y y^{-n}$

$\Rightarrow R_0 = \pi_1\Big($  $\Big)$

$|c,\mu| = r_0\, y^{e_1}\, r_1\, y^{e_2} \text{———} y^{e_n}\, r_n$ $\quad (e_i \in \mathbb{Z})$.

1 st case: $\sum e_i \neq 0$

$\Rightarrow |c,\mu| \notin R_0$, so $|c,\mu| \neq 1$.

2nd case: $\sum e_i = 0$

$\Rightarrow (c,\mu) = (c',\mu')$ reduced cycle for 

and c closed path

$\Rightarrow |c,\mu| \neq 1$ in $R_0 \subset F(G,Y)$.

case of trees of groups + $\boxed{\text{Cor. 3}}$

∎

4) Universal cover of a graph of groups

<u>Goal</u>: given a graph of groups (G,Y), build a tree T on which $\pi_1(G,Y)$ acts with
- quotient Y
- vertex stabilizers given by the vertex groups of G
- edge " " " edge " "

Idea: As for amalgamated free products, shrink the orbits of the vertex groups in a Cayley graph of $\pi_1(G,Y)$.

Previous theorem on normal forms $\Rightarrow$ the resulting graph is a tree.

Data: - a graph of groups (G,Y)

- a spanning tree T of Y

- an orientation A of Y (i.e. the subset of positive edges)

$\pi := \pi_1(G,Y,T)$

Not: $\cdot$ $e(y) = \begin{cases} 1 & \text{if } y \notin A \\ 0 & \text{otherwize} \end{cases}$

and $|y|$ = the edge $\in \{y, \bar{y}\} \cap A$

$\cdot$ g_y = image of y in $\pi_1(G,Y,T) = \begin{cases} y & \text{if } y \notin T \\ 1 & \text{otherwize} \end{cases}$

Goal: build - a graph $\tilde{X} = \tilde{X}(G,Y,T)$;

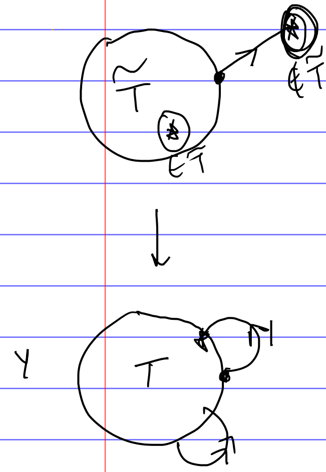
- an action $\pi \curvearrowright \tilde{X}$ s.t. $\pi \backslash \tilde{X} \simeq Y$;

- sections $Y \to \tilde{X}$ such that

$P \mapsto \tilde{P}$

$y \mapsto \tilde{y}$

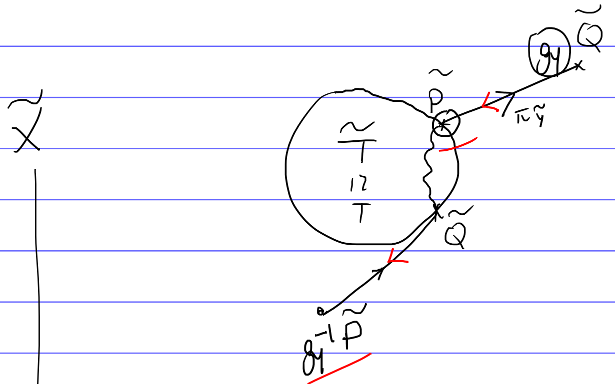
$o(\tilde{y}) = \widetilde{o(y)} \quad \forall y \in A$.

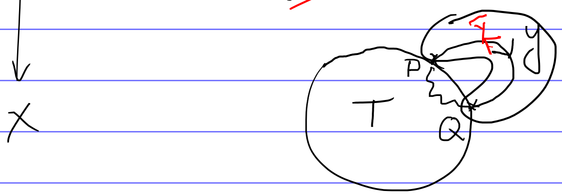


and $\begin{cases} \forall P \in V(Y), & \mathrm{Stab}_\pi(\tilde{P}) = G_P \\ \forall y \in E(Y), & \mathrm{Stab}_\pi(\tilde{y}) = G_{|y|}^{\overline{|y|}} \end{cases}$; $G_P := \pi_{\tilde{P}}$, $G_{|y|}^{\overline{|y|}} := \pi_{\tilde{y}}$

The last condition implies that

$$\nu(\tilde{X}) = \coprod_{P \in \nu(Y)} \pi / \pi_{\tilde{P}} \qquad \mathcal{E}(\tilde{X}) = \coprod_{y \in \mathcal{E}(Y)} \pi / \pi_{\tilde{y}}$$





let $(y \geq 0)$ $y \leq 0$

- $\overline{\pi_{\tilde{y}}} = \pi_{\tilde{y}}$
- $o(\pi_{\tilde{y}}) = g_y^{-1} \tilde{P} = g_y^{-1} \widetilde{o(y)}$
- $t(\pi_{\tilde{y}}) = (g_y) \tilde{Q} = (g_y) \widetilde{t(y)}$

i.e.
$$\boxed{\begin{cases} \overline{g\pi_{\tilde{y}}} = g\pi_{\tilde{\bar{y}}} \\ o(g\pi_{\tilde{y}}) = g g_y^{-e(y)} \widetilde{o(y)} \\ t(g\pi_{\tilde{y}}) = g g_y^{1-e(y)} \widetilde{t(y)} \end{cases}} \qquad \forall g \in \pi$$

Well defined : $g\,\pi_{\tilde{y}} = g'\,\pi_{\tilde{y}} \Leftrightarrow g^{-1}g' \in G_y^{(y)}$

If $e(y) = 0 \underset{|y|=y}{\Rightarrow} g^{-1}g' \in G_y^{y} \leq G_{o(y)}$

$\Rightarrow g' G_{o(y)} = g\, G_{o(y)}$

and $g^{-1}g' \in G_y^{y} = g_y \underbrace{G_y^{y}}_{\leq G_{t(y)}} g_y^{-1}$
$\leq g_y G_{t(y)} g_y^{-1}$

$\Rightarrow g' g_y G_{t(y)} = g g_y G_{t(y)}$

If $e(y) = 1$: same computation

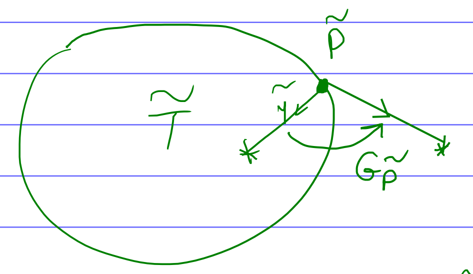
Theorem: $\tilde{X}$ is a tree.

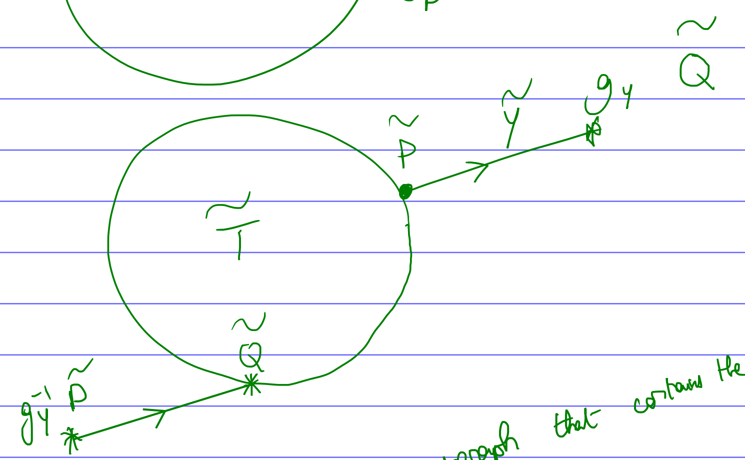
Proof:

- $\tilde{X}$ is connected: Denoting by

$S := \{ G_P,\ P \in V(Y),\ g_y,\ y \in E(Y) \}$ one has

$$\begin{cases} \pi = \langle S \rangle \\ s.\tilde{T} \cap \tilde{T} \neq \emptyset \quad \forall s \in S \end{cases}$$





smallest subgraph that contains the $\tilde{y}$.

$$\Rightarrow \tilde{X} = \bigcup_{g \in \pi} gW \text{ is connected.}$$

- $\tilde{X}$ has no circuit:

(ABS) Let $(s_1\tilde{y}_1, \dots, s_n\tilde{y}_n)$ be a circuit without backtracking in $\tilde{X}$. Note $\begin{cases} g_i := g_{y_i} \\ e_i := e(y_i) \end{cases}$

Obs: $(y_1, \dots, y_n)$ circuit in Y. Note: $P_i = t(y_i)$.

. $t(s_i\tilde{y}_i) = o(s_{i+1}\tilde{y}_{i+1}) \Rightarrow s_i g_i^{1-e_i} G_{P_i} = s_{i+1} g_{i+1}^{-e_{i+1}} G_{P_i}$

i.e. $s_{i+1} g_{i+1}^{-e_{i+1}} = s_i g_i^{1-e_i} r_i$, $r_i \in G_{P_i}$

i.e. $(q_i := s_i g_i^{-e_i})$ $\begin{cases} q_{i+1} = q_i g_i r_i & \forall i < n \\ q_1 = q_n g_n r_n \end{cases}$

$$\begin{aligned} \Rightarrow q_1 &= q_n g_n R_n \\ &= q_{n-1} g_{n-1} r_{n-1} g_n R_n \\ &\ \ \vdots \\ &= q_1 g_1 R_1 g_2 R_2 \cdots g_n R_n \end{aligned}$$

$\Rightarrow \underline{g_1 R_1 g_2 R_2 \cdots g_n R_n = 1}$.

Claim: The word $((y_1, \dots, y_n), (1, R_1, \dots, R_n))$ is reduced.

Proof of the claim: if $y_{i+1} = \overline{y_i}$, show that $r_i \notin G_{y_i}^{y_i}$.

$y_{i+1} = \overline{y_i} \Rightarrow \begin{cases} e_{i+1} = 1 - e_i \\ g_{i+1} = g_i^{-1} \end{cases}$

and $R_i = g_i^{e_i - 1} s_i^{-1} s_{i+1} g_{i+1}^{-e_{i+1}}$

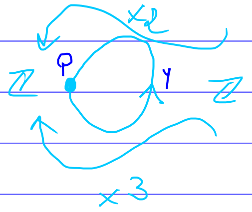
$= g_i^{e_i - 1} s_i^{-1} s_{i+1} g_i^{1-e_i}$

$\in G_{y_i}^{y_i}$ iff $s_i^{-1} s_{i+1} \in g_i^{1-e_i} G_{y_i}^{y_i} g_i^{e_i - 1}$

$= G_{\tilde{y}_i}^{\overline{|y_i|}}$

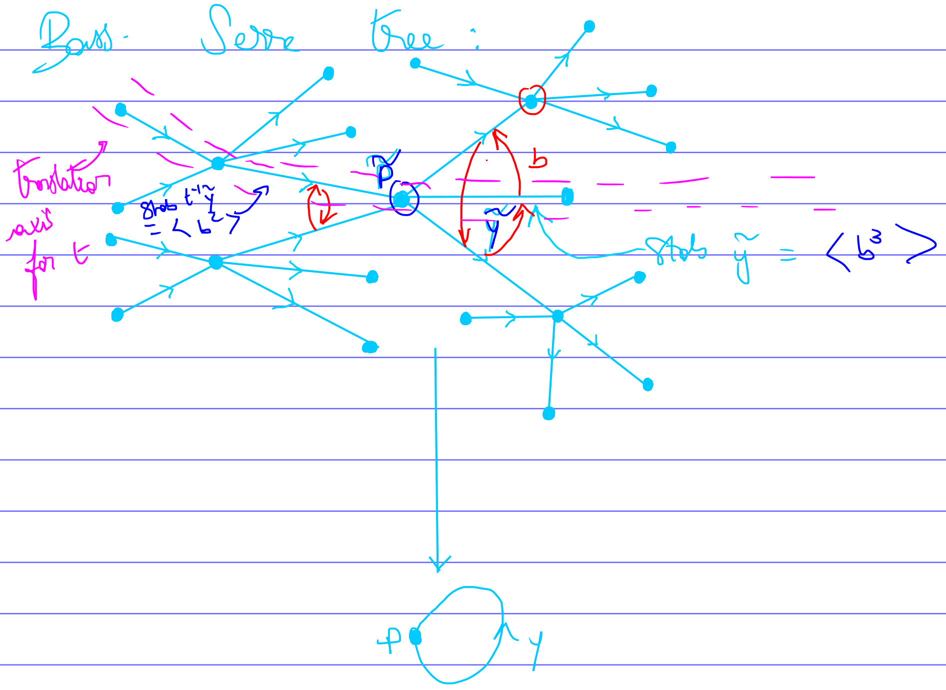
$= \pi_{\tilde{y}_i}$

thus $\overline{s_{i+1}\tilde{y}_{i+1}} = s_i\tilde{y}_i$, which contradicts the fact that the circuits is reduced ∎

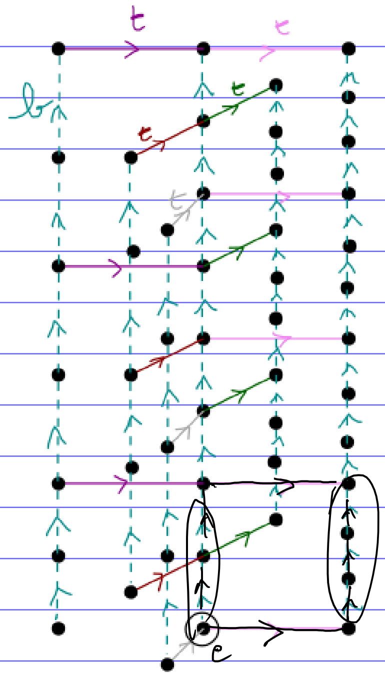
Example : 

$\langle b, t \mid t b^2 t^{-1} = b^3 \rangle$

Bass-Serre tree :



This tree is the Cayley graph of $BS(2,3)$ all of whose $\langle b \rangle$-orbits have been shrunk!



The fact that the quotient graph is a tree results from the theorem on normal forms.

5) Structure theorem

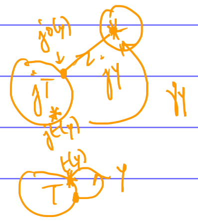
Let G be a group acting on a connected graph X.

Goal: if X is a tree, then G is the fundamental group of a graph of groups $(G, G\backslash X)$!

Construction of the graph of groups:

- $Y = G\backslash X$, T spanning tree of Y
- $j : T \to X$ a lift of T

Extended to Y as follows: $y \in A \setminus \mathcal{E}(T)$



$$\Rightarrow \exists!\ j_y \in \mathcal{E}(X) \text{ st } \begin{cases} G\cdot j_y = y \\ o(j_y) \in T \end{cases}$$

$$t(j_y) = \gamma_y\, j t(y)$$

Extend γ to $\mathcal{E}(Y)$ by setting $\gamma_{\bar{y}} = \gamma_y^{-1}$

And let $j_{\bar{y}} = \overline{j_y}$

In part:
$$\begin{cases} o(jy) = \gamma_y^{-e(y)}\, j o(y) \\ t(jy) = \gamma_y^{1-e(y)}\, j t(y) \end{cases}$$

Now define the vertex and edge groups:

- $\forall P \in V(Y)$, $G_P = \mathrm{Stab}(jP)$
- $\forall y \in \mathcal{E}(Y)$, $G_y = \mathrm{Stab}(jy)$

And the inclusions: $\forall y \in \mathcal{E}(Y)$

$$\underset{\mathrm{Stab}\, jy}{\overset{\shortparallel}{G_y}} \hookrightarrow G_{t(y)} = \mathrm{Stab}(j t y) = \mathrm{Stab}(\gamma_y^{e(y)-1} t jy) = \gamma_y^{e(y)-1} (\mathrm{Stab}\, t jy)\, \gamma_y^{1-e(y)}$$

$$a \mapsto \gamma_y^{e(y)-1}\, a\, \gamma_y^{1-e(y)}$$

Let $\varphi : \pi_1(G,Y,T) \longrightarrow G$

$a \in G_P \longmapsto a$

$g_y \longmapsto \gamma_y$

- φ is well-defined: $o_y \xrightarrow{y} t_y \qquad y \in A$

$$\Rightarrow \forall a \in G_y : \underline{\varphi(g_y)\varphi(a^y)\varphi(g_y^{-1})}$$
$$= \varphi(g_y)\,\varphi(g_y^{-1} a^{\bar y} g_y)\,\varphi(g_y^{-1})$$
$$= \gamma_y\,(\gamma_y^{-1} a^{\bar y} \gamma_y)\,\gamma_y^{-1}$$
$$= a^{\bar y}$$
$$= \varphi(a^{\bar y})$$

- Let $\tilde X$ be the Bass-Serre tree of $\pi_1(G,Y,T)$

and let $\Psi : \tilde X \longrightarrow X$

$g\tilde P \longmapsto \varphi(g)\, jP$

$g\tilde y \longmapsto \varphi(g)\, jy$

Goal:

⇝ φ is surjective: let $W = \underbrace{jY}_{\text{smallest subgraph of } X \text{ that contains } jT}$, $\Gamma = \mathrm{Im}\,\varphi$.

show that $\boxed{\Gamma \cdot W = X}$

if this is the case: let $g \in G$, $P \in V(Y)$ ($\Rightarrow jP \in jT$).

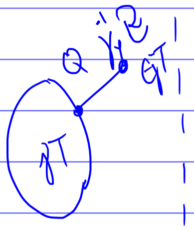
Then $\exists \gamma \in \Gamma$, $g\,jP = \gamma Q$

for some $Q \in W$.

$Q \in jT \Rightarrow jP = Q$

$\Rightarrow g^{-1}\gamma \in G_P \subseteq \Gamma$

$\Rightarrow g \in \Gamma$

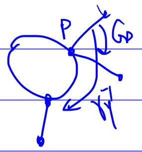


$Q \notin jT \Rightarrow \exists y,\ \gamma_y Q \in jT$

$\Rightarrow P = \gamma_y Q$

and $\gamma_y^{-1}\gamma^{-1} g \in G_P \subseteq \Gamma$

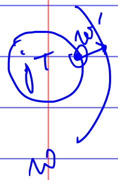
$\Rightarrow g \in G_P$



$\forall y \in G_P, \quad \gamma W \cap W \neq \emptyset$
$\gamma = \gamma_y$

$\Rightarrow \Gamma \cdot W$ is a connected subgraph of X

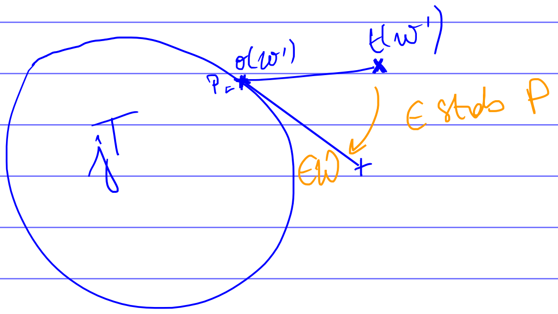
Assume $\exists$ edge w st $o\,w \in \Gamma \cdot W$ and show that $w \in \Gamma \cdot W$.



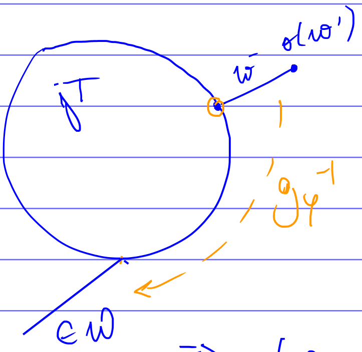
$\Rightarrow \exists \gamma \in \Gamma, \quad \gamma\, o\, w \in W$
$w' := \gamma w$

If $w' \notin jT$:

$o(w') \in T_0$



$o(w') \notin T$



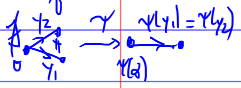
$\Rightarrow o(g_y w') \in T$

So $\Gamma \cdot W = X \Rightarrow \varphi$ is surjective

$\rightsquigarrow \psi$ is surjective;

$\rightsquigarrow \psi$ is locally injective: $y_1, y_2 \geq 0$

$\psi: G_1 \to G_2$ graph morphism
ψ loc. injective iff



$\psi(g_1 \tilde{y}_1) = \psi(g_2 \tilde{y}_2)$
wt $o(g_1 \tilde{y}_1) = o(g_2 \tilde{y}_2)$

$\Rightarrow \varphi(g_1)\, j y_1 = \varphi(g_2)\, j y_2$

$\Rightarrow \begin{cases} y_1 = y_2 := y \\ \varphi(g_2^{-1} g_1)\, j y = j y \end{cases}$

and $o(g_1 \tilde{y}_1) = o(g_2 \tilde{y}_2) \Rightarrow g_1^{-1} g_2 \in G_{o(y)}$

$\Rightarrow \varphi(g_1^{-1} g_2) = g_1^{-1} g_2$

Thus $g_1^{-1} g_2 \tilde{j}_Y = \tilde{j}_Y$

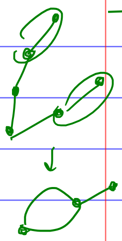
$\Rightarrow g_1 \tilde{Y}_1 = g_2 \tilde{Y}_2$.

Theorem: TFAE

(1) X is a tree;

(2) $\Psi : \tilde{X}(G,Y,T) \longrightarrow X$ is an isomorphism;

(3) $\varphi : \pi_1(G,Y,T) \longrightarrow G$ is an isomorphism.



Proof (1) $\Rightarrow$ (2) Ψ locally injective + surjective in a tree $\Rightarrow$ Ψ isomorphism;

(2) $\Rightarrow$ (1) $\tilde{X}$ is a tree by the previous thm;

(3) $\Rightarrow$ (2) clear;

(2) $\Rightarrow$ (3) Let $g \in G$ st $\varphi(g) = 1$.
For every P, $g\tilde{P} = \tilde{P}$.
In particular, $g \in G_P$
$\Rightarrow \varphi(g) = g = 1$. ∎

To summarize:

$G = \pi_1(G,Y,T) \longmapsto$ tree $\tilde{X}(G,Y,T) \curvearrowleft G$

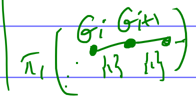
$G \curvearrowright T$ inherits a structure of graphs of groups $\longleftarrow G \curvearrowright T$

III / Some applications

1) Kurosh's theorem

Theorem: Let $G = \ast_{i \in I} G_i$ and let H be a subgroup of G. Then there exists
- a set J;
- a free subgroup F of G;
- $(g_j)_{j \in J} \in G^J$
- $\forall j \in J$, $H_j \leq G_{i_j}$ for some $i_j \in I$

st $\left[H = F \ast \ast_{j \in J} g_{i_j} H_j g_{i_j}^{-1} \right.$



Proof: • H acts on the Bass–Serre tree T of G with trivial edge stabilizers.

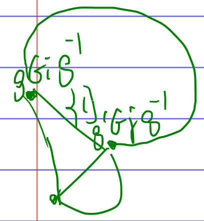
• The orbits of vertices in $H \backslash T$ are $\coprod_{i \in I} H \backslash G / G_i$

$\rightsquigarrow$ for every $i \in I$, let J_i be a set of representatives of $H \backslash G / G_i$, and set

$$J = \coprod_{i \in I} J_i$$

• for every $g_i \in J_i$:

$$\mathrm{Stab}_H (\underbrace{g_i G_i}_{\in T}) = H \cap \mathrm{Stab}_G\, g_i G_i$$
$$= H \cap g_i G_i g_i^{-1}$$



• The rank of F is the rank of the fundamental group of $H \backslash T$.

Example: Assume that I is finite, and that $\# H \backslash G / G_i < \infty$ $\forall i$, and that $[G : H] < \infty$

Then the quotient graph $H \backslash T$ has

$$s := \sum_i \# H \backslash G / G_i \text{ vertices}$$

$$a := \sum_{\substack{\text{edges of} \\ G \backslash T}} \# H \backslash G / \{1\} = (|I| - 1)\, [G : H] \text{ edges}$$

So by Euler's formula:

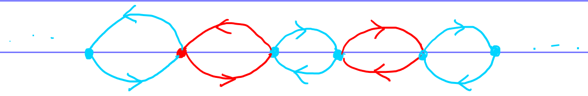
$$s - a + (\mathrm{rk}\, F + 1) = 2$$

$$\Rightarrow \mathrm{rk}(F) = 1 + (|I| - 1)[G : H] - \sum_i \# H \backslash G / G_i$$

2) Finite graphs of finite groups

Theorem: Let G be the fundamental group of a finite graph of finite groups. Then G is virtually free.

Example: $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$



Sketch of the proof: Find a finite set X and a morphism $\varphi : G \to \mathfrak{S}(X)$ such that $\ker \varphi$ intersects trivially every vertex group.

BS $\Rightarrow$ $\ker \varphi$ is free;
$\mathfrak{S}(X)$ finite $\Rightarrow$ $\ker \varphi$ has finite index in G.

Take $\#X = \prod_{G_i \text{ vertex group}} \#G_i$.

Obs: 1) For every finite group F and any finite set S, there exists a F-free action on S iff
In this case, it is unique. (i.e. the induced representations $F \to \mathfrak{S}(X)$ are conjugate)

2) If $F' \leq F$ and $\#F \mid \#S$, any free action of F' on S extends to a free action of F on X.

Construction of φ: fix a spanning tree T.

$\to$ define φ inductively on the vertex groups so that $\varphi_{o(e)}|_{G_e} = \varphi_{t(e)}|_{G_e}$ $\forall e \in T$ using the 2nd observation;

$\to$ $\varphi_{o(e)}|_{G_e}$ and $\varphi_{t(e)}|_{G_e}$ are conjugate by $\sigma_e \in \mathfrak{S}(X)$ for every $e \notin T$ (1st observation).

$\Rightarrow$ defines $\tilde{\varphi} : G \to \mathfrak{S}(X)$ st $\tilde{\varphi}|_{G_v} = \varphi|_{G_v}$, $t_e \mapsto \sigma_e$ $\forall e \notin T$ $\blacksquare$

3) Grushko's theorem

Theorem: Let A, B be finitely generated groups. Then $rk(A * B) = rk(A) + rk(B)$.

Sketch of the proof:

$\boxed{\leq}$ is clear.

$\boxed{\geq}$ • Let $n = rk(A * B)$. Let $(\gamma_1, \dots, \gamma_n)$ be a generating set of $A * B$, written in their normal forms.

$F = F_A * F_B$
$\varphi(F_A) = A$
$\varphi(F_B) = B$

• Let $F = F(\gamma_1, \dots, \gamma_n)$

$G = F(\text{letters in } \gamma_i \subseteq A) * F(\text{letters in } \gamma_i \subseteq B)$

• Let T_0 be the Cayley graph of G w.r.t this generating set.

• Let $\hat{T}$ be the Bass Serre tree of $A \xrightarrow[\{1\}]{} B$.

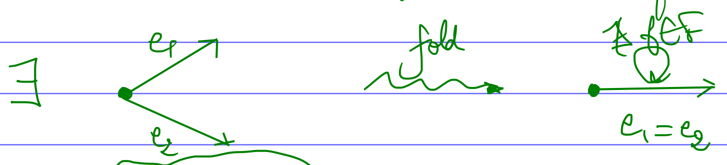
We will build a tree T endowed with

$\Rightarrow$ (i) an F-action with trivial edge stabilizers;

$\exists$ (ii) surjective F-equivariant morphisms $T_0 \to T \to \hat{T}$ st $F \backslash T = A*B \backslash \hat{T}$

The set of trees satisfying (i) contains T_0 and is inductive (for $T_1 \leq T_2 \iff \exists\, T_0 \to T_1 \to T_2 \to \hat{T}$). Let us show that a maximal element T is suitable.

If T has more than two F-orbits of edges:

$\exists$ 

st $Fe_1 \neq Fe_2$

which contradicts the maximality of T.

Thus $F = F_1 * F_2$ with $\begin{cases} \varphi(F_1) \leq A \\ \varphi(F_2) \leq B \end{cases}$

$\Rightarrow$ φ surj. $\begin{cases} A = \varphi(F_1) \\ B = \varphi(F_2) \end{cases} \Rightarrow \begin{cases} rk(A) \leq rk\, F_1 \\ rk(B) \leq rk\, F_2 \\ n = rk\, F = rk\, F_1 + rk\, F_2 \end{cases}$

$\Rightarrow rk(A) + rk(A) \leq n.$ ∎

Corollary: For any f.g group G, there exists a unique $r \geq 0$ and a unique decomposition

$G = G_1 * G_2 \_\_\_\_ * G_n * F$ (up to permutation and conjugation of the G_i's)

st

- F is a free group of rank r;
- G_i is non trivial, $\neq \mathbb{Z}$, and G_i is freely indecomposable $\forall i \in [1, n]$.

Proof: Existence: Grushko's theorem.

Uniqueness: Kurosh's theorem.